Algebraic sets with fully characteristic radicals

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Bu konuşma sevgili Kıvanç Ersoya ithaf olmuştur

Presentation Outline

First order theory of groups

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If all valid first order sentences of A and B are the same, then we can not conclude that A and B are isomorphic.

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5- left and right parentheses (and).

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"This is a finite group" can not be translated to a first order sentence.

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$$\mathrm{Th}_\forall(A)$$

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The solutions for all above problem was completed in 2006 by O. Kharlampovich, A. Miasnikov and Z. Sela (after fundamental works of Remeslennikov, Makanin, Razburov and many others)

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2- the solutions for the problem 2 are classified.
3- the universal theory of free groups is decidable.
4- a group A is universaly equivalent to a non-abelian free group if and only if it is fully residually free and they are limit points of the set of free groups in the space of marked groups.

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All three cases are almost the same, so we restrict here ourself to the case 1 (we don't use coefficients).

So, if $w(x_1, \ldots, x_n) \in F_n$, then $w \approx 1$ is a group equation. A solution to this equation is an element $(a_1, \ldots, a_n) \in A^n$ such that $w(a_1, \ldots, a_n) = 1$.

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Any intersection of algebraic sets is an algebraic set. The union is not so.

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Remark We have $Pvar(A) = SP(A)$, the least class containing A which is closed under product and subgroup.

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The class of equational noetherian groups is very large.

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Non-example: Baumslag-Solitar group

$$BS_{m,n} = \langle x, y \mid xy^m x^{-1} = y^n \rangle \quad (m, n > 1)$$

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5- Every subset of A^n is compact.

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Corollary Let $\mathbf{V}$ be a variety of groups which has finite axiomatic rank. If all elements of $\mathbf{V}$ are equational noetherian, then $\mathbf{V}$ is finitely based.

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Theorem Let A be a group and $\mathbf{V} = \mathrm{Var}_A(A)$ be the variety generated by A. Assume that for all $m \geq 1$, all finitely generated subgroup of A^m have $\max - n$. Assume also for all n, the group $F_{\mathbf{V}}(n)$ is finitely cogenerated. Then A is equational noetherian.

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Theorem Let A be a locally finite group and $\mathbf{V} = \mathrm{Var}_A(A)$ be the variety generated by A. Assume also for all n, the group $F_{\mathbf{V}}(n)$ is finitely cogenerated. Then A is equational noetherian.

Coordinate group and Unification

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Two categories of algebraic sets in A and coordinate groups over A are anti-isomorphic.

Determining the structures of coordinate groups is one of the main problems of theory. For equational noetherian groups we have the following Unification Theorem (Baumslag, Remeslennikov, Miasnikov)

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There are some other unification theorems which describe the coordinate groups in non-equational noetherian case.

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Theorem(Baumslag, Remeslennikov, Miasnikov)
Let A be an algebraically closed group. Then every finite system of equations over A satisfies nullestellensatz.

So this is the group-theoretic version of Hilbert's classical nullestellensatz.

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2- What is the necessary and sufficient condition for $\mathrm{Rad}_A(S)$ to be a fully characteristic subgroup of F_n?

Remark Fully characteristic means invariant under all endomorphisms. In free groups these are exactly the verbal subgroups.

Theorem 1 $\mathrm{Rad}_A(S)$ is fully characteristic if and only if $V_A(S) = \bigcup_i K_i^n$, where every K_i is an n-generator subgroup of A.

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Theorem 3 Let A be nilpotent of class at most n. Then $\mathrm{Rad}_A(S)$ is characteristic if and only if $V_A(S) = \bigcup_i K_i^n$, where every K_i is an n-generator subgroup of A.

An application of Theorem 1

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Theorem 4 Let S be a system of equations such that $V_A(S) = \bigcup_i K_i^n$ for some family of n-generator subgroups of A. Then there is a variety $\mathbf{V}$ such that $\Gamma_A(S) = F_{\mathbf{V}}(n)$. The converse is also true.

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One more application

Theorem 5 Let A be a equational noetherian group and W be a set of group words such that the verbal subgroup $W(F_n)$ is the radical of some subset of A^n. Let $\mathbf{V}$ be the variety of groups defined by W. Then $F_{\mathbf{V}}(n)$ is equational noetherian.