# Recognize some structural properties of a finite group from the orders of its elements

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Let $G$ be a periodic group.

*Basic Problem*

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*Obtain information about the structure of $G$ by looking at the orders of its elements.*

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- $\omega(G) = \{1, 2\}$ if and only if $G$ is an elementary abelian 2-group.
- If $\omega(G) = \{1, 3\}$, then $G$ is nilpotent of class $\leqslant 3$ (**F. Levi, B.L. van der Waerden, 1932**).
- If $\omega(G) = \{1, 2, 3\}$, then $G$ is (elementary abelian)-by-(prime order) (**B.H. Neumann, 1937**)

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If $\omega(G) \subseteq \{1, 2, 3, 4\}$, then $G$ is locally finite

(**I. N. Sanov, 1940**).

If $\omega(G) = \{1, 2, 3, 4\}$ ,

then $G$ is either an extension of an (elementary abelian 3-group) by (a cyclic or a quaternion group),

or

$G$ is an extension of a (nilpotent of class 2 2-group) by (a subgroup of $S_3$). (**D.V. Lytkina, 2007**)

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**Problem**

*Does $\omega(G) = \{1, 5\}$ imply $G$ locally finite?*

Still open

# Background- 1st. Direction

If $\omega(G) \subseteq \{1, 2, 3, 4, 5\}$, $\omega(G) \neq \{1, 5\}$ ,

then $G$ is locally finite.

(**N. D. Gupta, V.D. Mazurov, A.K. Zhurtov, E. Jabara, 2004**)

**Problem**

*Does* $\omega(G) = \{1,2,3,4,5,6\}$ *imply* $G$ *locally finite?*

Still open

If $G$ is a finite simple group, $G_1$ a finite group, $|G| = |G_1|$ and $\omega(G) = \omega(G_1)$ ,

then $G \simeq G_1$.

(M. C. Xu, W.J. Shi , 2003),
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# Background- 2nd Direction

$G$ a finite group

$e$ divisor of the order of $G$.

Write

$$L_e(G) := \{x \in G \mid x^e = 1\}.$$

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Obtain information about the structure of $G$

by looking at the orders of the sets $L_e(G)$.

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- $|L_e(G)|$ divides $|G|$, for every $e$ dividing $|G|$ (**Frobenius**)
- $|L_e(G)| = 1$, for every $e$ dividing $|G|$, if and only if $G$ is cyclic.

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# Background- 2nd Direction

**W. Meng and J. Shi**, in 2011, studied groups $G$ such that

$$|L_e(G)| \leq 2e, \text{ for every } e \text{ dividing } |G|.$$

**H. Heineken and F. Russo**, in 2015, studied groups $G$ such that

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# Background- 2ndDirection

**Problem**

*$G$ a soluble group, $G_1$ a finite group.*

*Does $|L_e(G)| = |L_e(G_1)|$,*
*for any $e$ dividing $|G|$, imply $G_1$ soluble?*

*(**J.G. Thompson**)*

Still open

Problem

Study some functions on the orders of the elements of $G$.

$G$ a finite group.

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$$\psi(G) := \sum_{x \in G} o(x)$$

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# Sum of the orders of the elements

Write $C_n$ the cyclic group of order n.

## Examples

$$\psi(S_3) = 13.$$

In fact we have $\psi(S_3) = 1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3.$

$$\psi(C_6) = 21.$$

In fact we have $\psi(C_6) = 1 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 2 \cdot 6.$

$$\psi(C_5) = 21.$$

In fact we have $\psi(C_5) = 1 \cdot 1 + 4 \cdot 5.$

# Sum of the orders of the elements

**Remark**

*$\psi(G) = \psi(G_1)$ does not imply $G \simeq G_1$.*

Write $A = C_6 \times C_2$,
$B = C_2 \ltimes C_6$, where $C_2 = \langle a \rangle$, $C_6 = \langle b \rangle$, $b^a = b^5$.
Then

$$\psi(A) = \psi(B) = 87.$$

**Remark**

*$|G| = |G_1|$ and $\psi(G) = \psi(G_1)$ do not imply $G \simeq G_1$.*

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# Sum of the orders of the elements

**Remark**

$\psi(G) = \psi(S_3)$ *implies* $G \simeq S_3$.

**Problem**

*Find information about the structure of a finite group $G$ from some inequalities on $\psi(G)$.*

# Sum of the orders of the elements

**Remark**

$\psi(G) = \psi(S_3)$ *implies* $G \simeq S_3$.

**Problem**

*Find information about the structure of a finite group* $G$ *from some inequalities on* $\psi(G)$.

# Sum on the orders of the elements

**Proposition**

*If $G = G_1 \times G_2$, where $|G_1|$ and $|G_2|$ are coprime, then*

$$\psi(G) = \psi(G_1)\psi(G_2).$$

# Sum of the orders of the elements in a cyclic group

### Remark

$\psi(C_n) = \sum_{d|n} d\varphi(d)$, *where* $\varphi$ *is the Eulero's function*

### Proposition

*Let* $p$ *be a prime. Then:*

$$\psi(C_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}.$$

# Sum of the orders of the elements in a cyclic group

## Proposition

*Let $p$ be a prime. Then:*

$$\psi(C_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1}.$$

*Proof.* $\psi(C_{p^\alpha}) = 1 + p\varphi(p) + p^2\varphi(p^2) + \cdots + p^\alpha(\varphi(p^\alpha)) =$

$1 + p(p-1) + p^2(p^2 - p) + \cdots + p^\alpha(p^\alpha - p^{\alpha-1}) =$

$= 1 + p^2 - p + p^4 - p^3 + \cdots + p^{2\alpha} - p^{2\alpha-1}) = \frac{p^{2\alpha+1}+1}{p+1}$, as required.//

## Corollary

*Let $n > 1$. Write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, $p_i'$s different primes. Then*

$$\psi(C_n) = \prod_{i\in\{1,\cdots,s\}} \frac{p_i^{2\alpha_i+1}+1}{p_i+1}.$$

# Sum of the orders of the elements

**Theorem (H. Amiri, S.M. Jafarian Amiri, M. Isaacs, Comm. Algebra 2009)**

*Let $G$ be a finite group, $|G| = n$. Then*

$$\psi(G) \leq \psi(C_n).$$

*Moreover*

$$\psi(G) = \psi(C_n) \text{ if and only if } G \simeq C_n.$$

# Sum of the orders of the elements

**Theorem ((1) , M. Herzog, P. Longobardi, M. Maj)**

*Let $G$ be a finite group, $|G| = n$, $q$ the minimum prime dividing $n$.*

*If $G$ is non-cyclic, then*

$$\psi(G) < \frac{1}{q-1}\psi(C_n).$$

Hence

$|G| = n$, $q$ the minimum prime dividing $n$. Then:

$\psi(G) \geq \frac{1}{q-1}\psi(C_n)$ implies $G$ cyclic.

# Sum of the orders of the elements

**Remark**

*It is not possible to substitute $q-1$ by $q$. In fact:*

$$\psi(S_3) = 13 \geq \frac{1}{2}\psi(C_6) = \frac{21}{2}.$$

**Theorem ((2) , M. Herzog, P. Longobardi, M. Maj)**

*Let $G$ be a finite group, $|G| = n$, $q$ the minimum prime dividing $n$.*

*If $\psi(G) \geq \frac{1}{q}\psi(C_n)$, then*

*$G$ is soluble and $G'' \leq Z(G)$.*

# Sum of the orders of the elements

**Theorem ((3) , M. Herzog, P. Longobardi, M. Maj)**

*Let $G$ be a finite group, $|G| = n$.*

*If $\psi(G) \geq n\varphi(n)$, then*

*$G$ is soluble and $G'' \leq Z(G)$.*

# Ideas of the proofs

**Lemma (1)**

*Let $G$ be a finite group, $p$ a prime,*
*$P$ a cyclic normal*
*$p$-Sylow subgroup of $G$. Then:*

$$\psi(G) \leq \psi(G/P)\psi(P).$$

*If $P \leq Z(G)$, then $\psi(G) = \psi(G/P)\psi(P)$.*

# Ideas of the proofs

**Lemma (2)**

*Let $n$ be a positive integer, $p$ the maximal prime dividing $n$, $q$ the minimum prime dividing $n$. Then:*

$$\varphi(n) \geq \frac{n}{p}(q-1).$$

# Proof of Theorem 3

- Assume $G = n$, $\psi(G) \geq n\varphi(n)$. First we show that $G$ is soluble.
- Write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, $p_1 > \cdots > p_s$ primes.
- Then $\psi(G) \geq \frac{n^2}{p_1}$, by Lemma 2. Then there exists an element $x$ of order $\geq \frac{n}{p_1}$.
- Thus $|G : \langle x \rangle| < p_1$.
- Then $G$ has a cyclic normal $p_1$-subgroup $P_1$.
- $G = P_1 \rtimes H$.

# Proof of Theorem 3

- Suppose $G = P_1 \rtimes P_2 \rtimes \cdots \rtimes P_t \rtimes K$, $P_i$ cyclic, $|K| = k$.
- $\psi(K) \geq k\varphi(k) \prod_{i=1}^{t} \frac{p_i^2 - 1}{p_i^2 + 1}$.

# Proof of Theorem 3

- Suppose $G = P_1 \rtimes P_2 \rtimes \cdots \rtimes P_t \rtimes K$, $P_i$ cyclic, $|K| = k$.
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**Lemma (Ramanujan 1913-1914)**

*Let $q_1, q_2, \cdots q_s, \cdots$ be the sequence of all primes:*

$$q_1 < q_2 < \cdots < q_s < \cdots .$$

*Then* $\prod_{i=1}^{\infty} \frac{q_i^2+1}{q_i^2-1} \quad = \quad \frac{5}{2}.$

## Lemma (3)

*Let $G$ be a finite group and suppose that there exists $x \in G$ such that*

$$|G : \langle x \rangle| < 2p,$$

*where $p$ is the maximal prime dividing $|G|$.*
*Then one of the following holds:*
*(i) $G$ has a cyclic normal $p$-subgroup,*
*(ii) $\langle x \rangle$ is a maximal subgroup of $G$, and $G$ is metabelian.*

# Proof of Theorem 3

- $\prod_{i=1}^{t} \frac{p_i^2-1}{p_i^2+1} \geq \frac{2}{3}$.
- Then $\psi(K) \geq k\varphi(k) \prod_{i=1}^{t} \frac{p_i^2-1}{p_i^2+1} \geq \ k\varphi(k)\frac{2}{3} \geq \frac{k^2}{p_{t+1}}\frac{2}{3}$.
- Then there exists an element $v \in K$ of order $\geq \frac{2}{3}\frac{k}{p_{t+1}}$.
- $|K : \langle v \rangle| < 2p_{t+1}$.
- By Lemma 3, either $\langle v \rangle$ is maximal in $K$ and $K$ is metabelian, or there exists a normal cyclic $p_{t+1}$-Sylow subgroup of $K$, and we can write $G = P_1 \rtimes P_2 \rtimes \cdots \rtimes P_{t+1} \rtimes L$, where $P_i$ is cyclic, for all $i$ .
- Continuing in this way, we get that either $G$ soluble, or
- $G = P_1 \rtimes P_2 \rtimes \cdots \rtimes P_s$, where $P_i$ is cyclic, for all $i$.
- In any case $G$ is soluble.

# Sum of the orders of the elements

Let $n$ be a positive integer. Put

$$T := \{\psi(H) \mid |H| = n\}$$

Recall that

$\psi(C_n)$ is the maximum of $T$.

**Problem**

*What is the structure of $G$ if $\psi(G)$ is the minimum of $T$?*

# Sum of the orders of the elements

$G$ is non-nilpotent.

**Theorem (H. Amiri, S.M. Jafarian Amiri, J. Algebra Appl, 2011)**

*Let $G$ be a finite nilpotent group of order $n$.*

*Then there exists a non-nilpotent group $K$ of order $n$ such that*

$$\psi(K) < \psi(G).$$

# Sum of the orders of the elements

**Conjecture (H. Amiri, S.M. Jafarian Amiri, J. Algebra Appl, 2011)**

*Let $G$ be a finite non-simple group, $S$ a simple group, $|G| = |S|$.*

*Then*

$$\psi(S) < \psi(G).$$

# Sum of the orders of the elements

**Theorem (S.M. Jafarian Amiri, Int. J. Group Theory, 2013)**

*Let $G$ be a finite non-simple group.*

*If $|G| = 60$, then $\psi(A_5) < \psi(G)$.*

*If $|G| = 168$, then $\psi(PSL(2,7)) < \psi(G)$.*

# Sum of the orders of the elements

Assume $G$ is a finite non-simple group.

Using GAP it is possible to see that:

if $|G| = 360$, then $\psi(A_6) < \psi(G)$.

if $|G| = 504$, then $\psi(PSL(2,8)) < \psi(G)$.

if $|G| = 660$, then $\psi(PSL(2,11)) < \psi(G)$.

# Sum of the orders of the elements

But the conjecture is not true.

**Theorem** (**Y. Marefat, A. Iranmanesh, A. Tehranian, J. Algebra Appl., 2013**)

*Let* $S = SL(2, 64)$ *and* $G = 3^2 \times Sz(8)$.

*Then* $\psi(G) \leq \psi(S)$.

# Sum of the orders of the elements

**Conjecture**

*Let $G$ be a finite soluble group, $S$ a simple group, $|G| = |S|$.*

*Then*

$$\psi(S) < \psi(G).$$

# Some other functions

Let $G$ be a finite group. Define:

$$P(G) := \prod_{x \in G} o(x)$$

## Some other functions

**Theorem (M. Garonzi, M. Patassini )**

*Let $G$ be a finite group, $|G| = n$. Then*

$$P(G) \leq P(C_n).$$

*Moreover*

$$P(G) = P(C_n) \text{ if and only if } G \simeq C_n.$$

# Some other functions

$G$ a finite group, $r, s$ real numbers. Define:

$$R_G(s,r) = \sum_{x \in G} \frac{o(x)^r}{\varphi(o(x))^s},$$

$$R_G(r) = R_G(r,r).$$

**Remark**

$$R_G(0,1) = \psi(G).$$

# Some other functions

**Theorem (M. Garonzi, M. Patassini )**

*Let $G$ be a finite group, $|G| = n$, $r < 0$.*

*Then $R_G(r) \geq R_{C_n}(r)$.*

*Moreover*

*If $R_G(r) = R_{C_n}(r)$, then $G$ is nilpotent.*

# Some other functions

**Problem**

*Let $G$ be a finite group, $r, s$ real numbers.*

*Does $R_G(r,s) = R_{C_n}(r,s)$ imply $G$ soluble?*

# Some other functions

**Theorem (T. De Medts, M. Tarnauceanu, 2008)**

*Let $G$ be a finite group.*

*If $G$ is nilpotent, then $R_G(1,1) = R_{C_n}(1,1)$.*

**Problem**

*Let $G$ be a finite group, $r, s$ real numbers.*

*Does $R_G(1,1) = R_{C_n}(1,1)$ imply $G$ nilpotent?*

# Some other functions

**Problem**

*Let $G$ be a finite group.*

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**Problem**

*Let $G$ be a finite group.*

*Does $\sum_{x \in G} \frac{o(x)}{\varphi(o(x))} \leq \sum_{x \in C_n} \frac{o(x)}{\varphi(o(x))}$?*

*Thank you for the attention* !

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