On the subgroup generated by autocommutators

Patrizia LONGOBARDI

UNIVERSITÀ DEGLI STUDI DI SALERNO

4th Cemal Koç Algebra Days

Middle East Technical University, Ankara, Turkey

22-23 April 2016

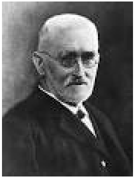
Dedication

Dedicated to the memory of **Cemal Koç**

Let G be a group, $x, y \in G$.
The **commutator** of x and y is the element

$$[x, y] := x^{-1}y^{-1}xy = x^{-1}x^{y}.$$

1896



Julius Wilhelm Richard **Dedekind**
1831 - 1916

Ferdinand Georg **Frobenius**
1849 - 1917

Some history

Results proved by **Dedekind** in 1880

The conjugate of a commutator is again a commutator.

Therefore the **commutator subgroup** *generated by the commutators of a group is a normal subgroup of the group.*

Any normal subgroup with abelian quotient contains the commutator subgroup.

The commutator subgroup is trivial if and only if the group is abelian.

First published by **G.A. Miller** in 1896

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Dedekind had studied normal extensions of the rational field with all subfields normal. Some years later these investigations suggested to him the related problem:

Characterize those groups with the property that all subgroups are normal.

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George Abram **Miller**
1863 - 1951

Heinrich Martin **Weber**
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"On the operation $sts^{-1}t^{-1}$"

The label commutator is used in

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(where the author expands the basic properties of the commutator subgroup and introduces the derived series of a group; he also shows that the derived series is finite and ends with 1 if and only if the group is solvable)

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The first textbook to introduce commutators and the commutator subgroup is **Weber**'s 1899 Lehrbuch der Algebra

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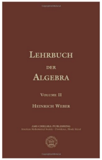

Lehrbuch der Algebra
1895 - 1896

the last important textbook on algebra published in the nineteenth century.

Some history

The first explicit statement of the

Question

Is the set of all commutators a subgroup?
i.e. Does the commutator subgroup consist entirely of commutators?

is found in Weber's 1899 textbook.
He states that

the set of commutators is not necessarily a subgroup.

In Miller's 1899 paper it is proved that the answer to the question is **yes** in the alternating group on n letters, $n \geq 5$, and in the holomorph of a finite cyclic group.

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The first example of a group in which the set of commutators in not equal to the commutator subgroup appears in

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Metabelian = Nilpotent of class ≤ 2

Fite constructs an example G of order 1024, attributed to Miller, then provides a homomorphic image H of order 256 of G which is again an example.

H is the subgroup of S_{16}:

$$H =< (1,3)(5,7)(9,11), (1,2)(3,4)(13,15),$$

$$(5,6)(7,8)(13,14)(15,16), (9,10)(11,12) >$$

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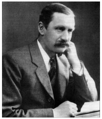
Some history

In

W. Burnside, On the arithmetical theorem connected with roots of unity and its application to group characteristics, *Proc. LMS* **1** (1903), 112-116

Burnside uses characters to obtain a criterion for when an element of the commutator subgroup is the product of two or more commutators.

William Benjamin **Fite**
1869 - 1932



William **Burnside**
1852 - 1927

Some history

The first occurence of the commutator notation probably is in

F.W. Levi, B.L. van der Waerden, Über eine besondere Klasse von Gruppen, *Abh. Math. Seminar der Universität Hamburg* **9** (1933), 154-158,

where the commutator of two group elements i, j is denoted by

$$(i, j) = iji^{-1}j^{-1}.$$

Friedrich Wilhelm **Levi**
1888 - 1966

Bartel Leendert **van der Waerden**
1903 - 1996

Some history

Hans Julius **Zassenhaus**
1912 - 1991

Lehrbuch der Gruppentheorie
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Philip **Hall**
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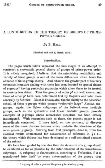

1932.] GROUPS OF PRIME-POWER ORDER. 29

A CONTRIBUTION TO THE THEORY OF GROUPS OF PRIME-POWER ORDER

By P. HALL.

[Received and read 10 March, 1933.]

Introduction.

The pages which follow represent the first stages of an attempt to construct a systematic general theory of groups of prime-power order. It is widely recognised, I believe, that the astonishing multiplicity and variety of these groups is one of the main difficulties which beset the advance of finite-group-theory. And hitherto the greater part of the very extensive literature dealing with them has been devoted to special classes of p-groups* having particular properties which allow them to be treated in more or less detail. Thus the groups of order p^4 are well known, and those of order p^5 have been determined first by Bagnera and later more concisely by Schreier. Much is known also, thanks chiefly to the American school, of those p-groups which possess "relatively large" Abelian subgroups. Again, the Sylow subgroups of the better-known insoluble groups, such as the symmetric and linear modular groups, provide examples of p-groups whose remarkable structure has been closely investigated. With researches such as these, the present paper is not immediately concerned. I have tried, on the contrary, to discover some of the more obvious features which underlie the structure of the most general p-group. Starting from first principles—that is, from the classical results enumerated for convenience of reference in § 1.1—the theory which follows has been developed as a self-contained account of the subject.

We have been guided by the idea that the structure of a group should be exhibited as far as possible by the inter-relations of its characteristic subgroups. A subgroup is characteristic, according to Frobenius, if it is transformed into itself by every automorphism of the group; thus

* A group of order a power of the prime p is called a p-group.

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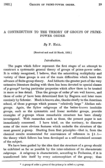

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Let G be a group and put

$$K(G) := \{[g,h] \,|\, g,h \in G\}.$$

Then

$$G' = < K(G) >.$$

Question

Is $G' = K(G)$?

When is $G' = K(G)$?

Which is the minimal order of a counterexample?

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There are exactly two nonisomorphic groups G of order 96 such that $K(G) \neq G'$. In both cases G' is nonabelian of order 32 and $|K(G)| = 29$.

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"On commutators in groups"

Groups St. Andrews 2005, Vol. 2, 531-558, London Math. Soc. Lecture Notes Ser., **340** , Cambridge University Press, 2007,

by

L-C. **Kappe**

R.F. **Morse**

Background

Many authors have considered subsets of a group G related to commutators asking if they are subgroups.

For instance, **W.P. Kappe** proved in 1961 that the set $R_2(G) = \{x \in G | \, [x, g, g] = 1, \forall g \in G\}$ of all right 2-Engel elements of a group G is always a subgroup.

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Basic definitions

Definition

Let G be a group, $g \in G$ and $\varphi \in Aut(G)$. The **autocommutator** of g and φ is the element

$$[g, \varphi] := g^{-1} g^{\varphi}.$$

We denote by

$$K^{\star}(G) := \{[g, \varphi] \mid g \in G, \varphi \in Aut(G)\}$$

the set of all autocommutators of G and, following P.V. Hegarty, we write

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A new problem

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Is $G^\star = K^\star(G)$? *Does it hold if G is abelian?*

At "Groups in Galway 2003" Desmond MacHale brought this problem to the attention of L-C. Kappe. He added that there might be an abelian counterexample and that perhaps the two groups of order 96 given by Guralnick as the minimal counterexamples to the conjecture $G' = K(G)$ might also be counterexamples.

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Results in the finite abelian case

Theorem (D. Garrison, L.-C. Kappe and D. Yull, 2006)

Let G be a finite abelian group. Then the set of autocommutators always forms a subgroup.

Furthermore there exists a finite nilpotent group of class 2 and of order 64 in which the set of all autocommutators does not form a subgroup. And this example is of minimal order.

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Esempio

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The abelian case

Let $(G, +)$ be an abelian group, $g \in G$ and $\varphi \in Aut(G)$. Then the **autocommutator** of g and φ is the element

$$[g, \varphi] := -g + g^{\varphi}.$$

Proposition

Let G be an abelian torsion group without elements of even order. Then

$$K^{\star}(G) = G^{\star} = G.$$

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The mapping $\tau : g \in G \longmapsto 2g \in G$ is an automorphism of G and $[g, \tau] = -g + g^{\tau} = g$. □

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- *If $B = \langle b_1 \rangle \oplus H$, with $|b_1| = 2^n$, $\exp H \leq 2^{n-1}$, then $K^\star(G) = G_{2^{n-1}} \oplus O$ where $G_{2^{n-1}} = \{x \in G \mid 2^{n-1}x = 0\}$.*

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In any abelian group G the map

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Example

Let $G = \langle a \rangle \oplus \langle c \rangle$, where $\langle a \rangle$ is infinite cyclic and $|c| = 2$. Then $K^\star(G)$ is not a subgroup of G.

Proof.

Let $\varphi \in Aut(G)$, then $\varphi(c) = c$, and $\varphi(a) = \gamma a + \delta c$, where $\gamma \in \{1, -1\}$ and $\delta \in \{0, 1\}$.
Therefore we have: $Aut(G) = \{1, \varphi_1, \varphi_2, \varphi_3\}$,
where $1 = id_G$, $\varphi_1(a) = -a$, $\varphi_1(c) = c$,
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Finitely generated infinite abelian groups

Theorem (L-C. Kappe, P.L., M. Maj)

Let G be a finitely generated infinite abelian group. Write

$$G = \langle a_1 \rangle \oplus \cdots \oplus \langle a_s \rangle \oplus B \oplus O,$$

where $a_1, \cdots, a_s$ are aperiodic, O is a finite group of odd order, B is a finite 2-group. Then we have:
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(iii) If $s = 1$ and $B = \langle b_1 \rangle \oplus H$, with $|b_1| = 2^n$, $expH \leq 2^{n-1}$, then $K^\star(G)$ is not a subgroup of G.
(iv) If $s = 0$ and either $B = 1$ or $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus H$, with $|b_1| = |b_2| = 2^n$, $expH \leq 2^n$, then $K^\star(G) = G^\star = G$. If $s = 0$ and $B = \langle b_1 \rangle \oplus H$, with $|b_1| = 2^n$, $expH \leq 2^{n-1}$, then $K^\star(G) = G_{2^{n-1}} \oplus O$ where $G_{2^{n-1}} = \{x \in G \mid 2^{n-1}x = 0\}$.
In any case, if $s = 0$, $K^\star(G)$ is a subgroup of G.

Periodic abelian groups

Theorem

Let G be a periodic abelian group.
Write $G = O \oplus D \oplus R$, where D is a divisible 2-group, R is a reduced 2-group and every element of O has odd order. Then

$$K^\star(G) = O \oplus D \oplus K^\star(R),$$

where

- $K^\star(R) = R$ *if R is of infinite exponent;*
- $K^\star(R) = R$ *if R is of finite exponent 2^n, and $R = \langle a \rangle \oplus \langle b \rangle \oplus H$, with $|a| = |b| = 2^n$;*
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Let G be a periodic abelian group.
Write $G = O \oplus D \oplus R$, where D is a divisible 2-group, R is a reduced 2-group and every element of O has odd order. Then

$$K^\star(G) \supseteq O \oplus D \oplus K^\star(R).$$

For, let $a \in O, b \in D, c \in K^\star(R)$, let $b = (-2v)$ for some $v \in D$ and let $\varphi \in Aut(R)$ such that $c = -t + t^\varphi$, for some $t \in R$. Consider the automorphism τ of G defined by putting $x^\tau = 2x$ for any $x \in O$, $y^\tau = -y$ for any $y \in D, r^\tau = r^\varphi$, for any $r \in R$.
Then $[a + v + t, \tau] = -a - v - t + (a + v + t)^\tau =$
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Lemma

Let R be a reduced abelian 2-group of infinite exponent. Then

$$K^\star(R) = R.$$

Proof - Sketch.

Let $g \in R$, and write $|g| = 2^n$. Then there exists $c \in R$ such that $|c| = 2^{n+1}$ and $R = \langle c \rangle \oplus H$, for some subgroup H of R. It is possible to show that $R = \langle c + g \rangle \oplus H$. Therefore there exixts an automorphism φ of R such that $c^\varphi = c + g, y^\varphi = y$ for any $y \in H$. Then $[c, \varphi] = -c + c^\varphi = g$, and $g \in K^\star(R)$, as required. $\square$

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Mixed abelian groups

Generalizing the previous example:

Example

Let $G = \langle a \rangle \oplus \langle c \rangle$, where $\langle a \rangle$ is infinite cyclic and $|c| = 2$. Then $K^{\star}(G)$ is not a subgroup of G.

it is easy to construct examples of mixed abelian groups G in which $K^{\star}(G)$ is not a subgroup. In fact, we have:

Proposition

Let T be a periodic abelian group with $K^{\star}(T) \subset T$ and consider the group $G = T \oplus \langle a \rangle$, where $\langle a \rangle$ is an infinite cyclic group. Then $K^{\star}(G)$ is not a subgroup.

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In the group G of the proposition the torsion subgroup $T(G) = T$ is contained in $K^{\star}(G)$, but $K^{\star}(T) \subset T$. Thus it is not true that $T \cap K^{\star}(G) \subseteq K^{\star}(T)$. Surprising, the reverse inclusion holds, in fact we have:

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Let G be a mixed abelian group and write $T = T(G)$ the torsion subgroup of G. Then

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Torsion-free abelian groups

Now consider torsion-free abelian groups.
Torsion-free abelian groups with a finite automorphim group have been studied by **de Vries** and **de Miranda** in 1958 and by **Hallett** and **Hirsch** in 1965 and 1970.

Theorem (J.T. Hallett, K.A. Hirsch)

If the finite group Γ is the automorphism group of a torsion-free abelian group A, then Γ is isomorphic to a subgroup of a finite direct product of groups of the following types:

(a) *cyclic groups of orders 2, 4, or 6;*
(b) *the quaternion group Q_8 of order 8;*
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Proposition

Let G be a group with cyclic automorphism group. Then $K^\star(G)$ is a subgroup of G.

Proof.

From $G/Z(G) \simeq Inn(G) \leq Aut(G)$, we get that G is abelian. If $|G| > 2$ the map $x \in G \longmapsto -x \in G \in Aut(G)$ has order 2, therefore $Aut(G)$ is also finite. Write $Aut(G) = \langle \varphi \rangle$ and put $|Aut(G)| = n$. The map $\theta : x \in G \longmapsto -x + x^\varphi \in G$ is a homomorphism of G. Therefore $Im\theta$ is a subgroup of G. Obviously $Im\theta \subseteq K^\star(G)$. We show that $K^\star(G) = Im\theta$ and then it is a subgroup of G. Let $s \in K^\star(G)$. Then $s = -x + x^{\varphi^i}$, for some $i \in \{1, \cdots, n-1\}$ and some $x \in G$. We have $-x + x^\varphi, -x^\varphi + x^{\varphi^2}, \cdots, -x^{\varphi^{i-1}} + x^{\varphi^i} \in Im\theta$, thus $-x + x^\varphi - x^\varphi + x^{\varphi^2} - \cdots - x^{\varphi^{i-1}} + x^{\varphi^i} = -x + x^{\varphi^i} = s \in Im\theta$. Therefore $K^\star(G) = Im\theta$, as required. □

Proposition

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There exist torsion-free abelian groups G of any rank with $Aut(G)$ of order 2, for them $K^\star(G) = 2G$ is a subgroup of G.

If G is a torsion-free abelian group of rank 1, then $G/2G$ has order at most 2, thus for any $x \in G$ and $\varphi \in Aut(G)$ we have $x^\varphi + 2G = x + 2G$, therefore $-x + x^\varphi \in 2G$ and $K^\star(G) = 2G$ is a subgroup of G.

de Vries and de Miranda and Hallett and Hirsch constructed many examples of abelian groups G, indecomposable or not, of rank ≥ 2, with $Aut(G) \simeq V_4$. In their examples $K^\star(G) = 2G$ is a subgroup of G.

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There exists a torsion-free abelian group of rank 2 such that $Aut(G) \simeq V_4$ and $K^\star(G)$ is not a subgroup of G.

Proposition

Let G be a torsion-free abelian group such that $Aut(G) \simeq Q_8$. If $G/2G$ has rank at most 4, then $K^\star(G)$ is a subgroup of G.

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Thank you for the attention !

P. Longobardi
Dipartimento di Matematica
Università di Salerno
via Giovanni Paolo II, 132, 84084 Fisciano (Salerno), Italy
E-mail address : plongobardi@unisa.it

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