Finite Groups admitting a Dihedral Group of Automorphisms

İsmail Ş. Güloğlu

Doğuş University, İstanbul

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The General Framework

G is a finite group,
A another finite group acting by automorphisms on G.
The elements of $C_G(A) = \{g \in G : g^a = g \text{ for any } a \in A\}$ are called fixed-points of A.
We say that A acts fixed-point-freely on G if the identity element og G is the only fixed-point of A on G.

The origin

J.G.Thompson (1959)

A finite group G having a fixedpointfree automorphism α of prime order is nilpotent.

What can we say about the structure of a group admitting a fixedpointfree group of automorphisms?

The results giving partial answers to the title question are (sometimes called Hall-Higman type) theorems

1. showing the solvability of G and theorems
2. bounding some group theoretical invariants of G in terms of some invariants of A.

Most of the the theorems are given under the hypothesis $(|G|, |A|) = 1$ Gross, Shult, Berger, Kurzweil, Hartley made important contributions culminating finally in the work of Turull .
The most important result without the coprimeness condition is due to Dade which gives an exponential bound for the nilpotent height of a solvable group G admitting a nilpotent fixed-point-free group A of automorphisms in terms of the number of primes dividing the order of A .
Other researchers in that direction Espuelas, Ercan, Guloglu

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- G is solvable if $(|G|, |A|) = 1$. (A consequence of CFSG)
- G is solvable if A is nilpotent. (Hartley-Belyaev)
- If A is any nonnilpotent group and H is any group then there exists a group G admitting A as a group of fixed-point-free automorphisms and having a quotient isomorphic to H.

Main Conjecture

Main Conjecture: If $(|G|, |A|) = 1$ or A is nilpotent, and if $C_G(A) = 1$ then nilpotent length of G is bounded by the length of the longest chain of subgroups of A.

Work of Turull

Work of Khukhro, Shumyatsky, Makarenko

Khukhro initiated the study of the case $A = FH$ is a Frobenius group with kernel F and complement H and $C_G(F) = 1$.
The dependence of certain invariants (such as the order, the rank, the nilpotent length, the nilpotency class and the exponent) of the group G on the corresponding invariants of $C_G(H)$ have been studied recently by Khukhro, Makarenko and Shumyatsky. As an example we state the following result of Khukhro (2012)

Theorem

Suppose that a group G admits a Frobenius group $A = FH$ of automorphims with kernel F and complement H such that $C_G(F) = 1$. Then

$$F_k(C_G(H)) = F_k(G) \cap C_G(H) \text{ for all } k.$$

To understand the very powerful theorem of Khukhro one tries to see how far one can weaken the hypothesis of this theorem.
In this direction the research conducted by Ercan and Guloglu show that similar results can be obtained by slightly weaker conditions:
$C_G(F) = 1$ can be replaced by $C_G(F)H$ is Frobenius with complement H
FH can be taken as Frobenius-like groups.

Shumyatsky : Dihedral groups

Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β and let $F = \langle \alpha\beta \rangle$. (Here, $D = FH$ where $H = \langle \alpha \rangle$) Suppose that D acts on the group G by automorphisms in such a way that $C_G(F) = 1$. If $C_G(\alpha)$ and $C_G(\beta)$ are both nilpotent then G is nilpotent.

Our Theorem

Let $D = \langle \alpha, \beta \rangle$ *be a dihedral group generated by the involutions* α *and* β *and let* $F = \langle \alpha\beta \rangle$. *Suppose that* D *acts on the group* G *by automorphisms in such a way that* $C_G(F) = 1$. *Then the nilpotent length of* G *is equal to the maximum of the nilpotent lengths of the subgroups* $C_G(\alpha)$ *and* $C_G(\beta)$.

Proof of the Theorem

Proposition *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β. Let $F = \langle \alpha\beta \rangle$ and $H = \langle \alpha \rangle$. Suppose that $D = FH$ acts on a q-group Q for some prime q and let V be a kQD-module for a field k of characteristic different from q such that the group F acts fixed point freely on the semidirect product VQ. If $C_Q(H)$ acts nontrivially on V then we have $C_V(H) \neq 0$ and*

$$Ker(C_Q(H) \, on \, C_V(H)) = Ker(C_Q(H) \, on \, V).$$

Counterexample to the proposition

We choose a counterexample with minimum $\dim_k V + |QD|$ and proceed over several steps.Let $K = Ker(C_Q(H)\, on\, C_V(H))$.

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- *(3) Let Ω denote the set of Q-homogeneous components of V. K acts trivially on every element W in Ω such that $Stab_H(W) = 1$ and so H fixes an element of Ω.*

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- *(4) F acts transitively on Ω.*

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- *(3) Let Ω denote the set of Q-homogeneous components of V. K acts trivially on every element W in Ω such that $Stab_H(W) = 1$ and so H fixes an element of Ω.*
- *(4) F acts transitively on Ω.*
- From now on W denotes an H-invariant element of Ω. Set $F_1 = Stab_F(W)$ and let T be a transversal for F_1 in F containing 1 . Then $F = \bigcup_{t \in T} F_1 t$ and so $V = \bigoplus_{t \in T} W^t$. Note that an H-orbit on $\Omega = \{W^t : t \in T\}$ is of length at most 2.

Counterexample to the proposition

- *(5) The number of H-invariant elements in Ω is at most 2 and is equal to 2 if and only if $|F/F_1|$ is even. Furthermore $V = U \oplus X$ where X is a Q-submodule centralized by K and U is the direct sum of all H-invariant elements in Ω.*

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- *(6) Since* $1 \neq K \trianglelefteq C_Q(H)$, *we can choose a nonidentity element* $z \in K \cap Z(C_Q(H))$. *Set* $L = \langle z \rangle$. *Then* $Q = L^{F_2} C_Q(U)$ *where* $F_2 = Stab_F(U)$.

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- *(7) Set $Y = F_{q'}$. Then $Y \cap F_1 \neq Y \cap F_2$.*

Suppose that $Y \cap F_1 = Y \cap F_2$. Pick a simple commutator $c = [z^{f_1}, \dots, z^{f_m}]$ of maximal weight in the elements $z^f, f \in F_1$ such that $c \notin C_Q(W)$. Since $Q = L^{F_2} C_Q(W)$, the weight of this commutator is equal to the nilpotency class of $Q/C_Q(W)$. It should be noted that the nilpotency classes of $Q/C_Q(W)$ and Q are the same, since Q can be embedded into the direct product of $Q/C_Q(W^f)$ as f runs through F. Hence $c \in Z(Q)$. Clearly, $C_Q(F) = 1$ implies $C_Q(Y) = 1$ and hence $\prod_{x \in Y} c^x = 1$. In fact we have

$$1 = \prod_{x \in Y} c^x = \prod_{x \in Y - F_1} c^x \prod_{x \in Y \cap F_1} c^x.$$

Recall that $[Z(Q), F_1] \leq \bigcap_{f \in F} C_Q(W^f) = C_Q(V) = 1$. This gives $\prod_{x \in Y \cap F_1} c^x = c^{|Y \cap F_1|}$. On the other hand, for any $f \in F_1$, $fx \notin F_2$ and so z centralizes $W^{(fx)^{-1}}$, that is, $z^{fx} \in C_Q(W)$. Therefore c^x lies in $C_Q(W)$ for any x in $Y - F_1$. It follows that $\prod_{x \in Y - F_1} c^x \in C_Q(W)$. This forces that $c^{|Y \cap F_1|} \in C_Q(W)$ which is impossible as $c \notin C_Q(W)$.

Counterexample to the proposition does not exist

Proof.

By (5) and (7), $|F_2 : F_1| = 2$ and q is odd. Now $Z_2(Q) = [Z_2(Q), H]C_{Z_2(Q)}(H)$ as $(|Q|, |H|) = 1$. We prove that Thus $[L, Z_2(Q)] \leq C_Q(U)$ Then we have $[L^{F_2}, Z_2(Q)] \leq C_Q(U)$, as U is F_2-invariant, which yields that $[Q, Z_2(Q)] \leq C_Q(U)$. Thus $[Q, Z_2(Q)] \leq \bigcap C_Q(U)^f = C_Q(V) = 1$ and hence Q is abelian. Now $[Q, F_1 H] \leq C_Q(W)$ due to the scalar action of $Q/C_Q(W)$ on W. Notice that $C_W(H) = 0$ because otherwise K is trivial on W due to its action by scalars. So H inverts every element of W and hence W^t. That is, H acts by scalars and hence lies in the center of $QF_2H/C_{QF_2}(U)$. On the other hand H inverts $F_2/C_{F_2}(U)$. It follows that $|F_2/C_{F_2}(U)| = 1$ or 2. Since $|F_2 : F_1| = 2$, we have $F_1 \leq C_{F_2}(U)$. This contradicts the fact that $C_W(F_1) = 0$ as $C_V(F) = 0$. □

Suppose that $n = f(G) \geq f(C_G(\alpha)) \geq f(C_G(\beta))$ and set $H = \langle \alpha \rangle$. We may assume that $[G, F] = G$, since $C_G(F) = 1$. For each prime p dividing $|G|$ there is a unique D-invariant Sylow p-subgroup of G. This yields the existence of an irreducible D-tower,i.e. a sequence of subgroups $\hat{P}_1, \ldots, \hat{P}_n$ such that

(a) $\hat{P}_i$ *is a D-invariant* p_i*-subgroup,* p_i *is a prime,* $p_i \neq p_{i+1}$*,*
for $i = 1, \ldots, n-1$*;*
(b) $\hat{P}_i \leq N_G(\hat{P}_j)$ *whenever* $i \leq j$*;*
(c) $P_n = \hat{P}_n$ *and* $P_i = \hat{P}_i / C_{\hat{P}_i}(P_{i+1})$ *for* $i = 1, \ldots, n-1$
and $P_i \neq 1$ *for* $i = 1, \ldots, n$*;*
(d) $\Phi(\Phi(P_i)) = 1$*,* $\Phi(P_i) \leq Z(P_i)$*, and* $exp(P_i) = p_i$ *when* p_i *is odd*
for $i = 1, \ldots, n$*;*
(e) $[\Phi(P_{i+1}), P_i] = 1$ *and* $[P_{i+1}, P_i] = P_{i+1}$ *for* $i = 1, \ldots, n-1$*;*
(f) $(\Pi_{j<i}\hat{P}_j)FH$ *acts irreducibly on* $P_i/\Phi(P_i)$ *for* $i = 1, \ldots, n$*;*
(g) $P_1 = [P_1, F]$

Set now $X = \prod_{i=1}^{n} \hat{P}_i$. As $P_1 = [P_1, D]$ by *(g)*, we observe that $X = [X, D]$. If X is proper in G, by induction we have $n = f(X) = f(C_X(H))$ and so the theorem follows. Hence $X = G$. Notice that G is nonabelian and hence $C_G(H) \neq 1$, that is $f(C_G(H) \geq 1$. Therefore the theorem is true if $G = F(G)$. We set next $\overline{G} = G/F(G)$. As $\overline{G}$ is a nontrivial group such that $\overline{G} = [\overline{G}, F]$, it follows by induction that $f(\overline{G}) = n - 1 = f(C_{\overline{G}}(H))$. This yields that $[C_{\overline{\hat{P}_{n-1}}}(H), \ldots, C_{\overline{\hat{P}_1}}(H)]$ is nontrivial. Since $C_{\overline{\hat{P}_i}}(H) = \overline{C_{\hat{P}_i}(H)}$ for each i by Lemma (ii), we have $Y = [C_{\hat{P}_{n-1}}(H), \ldots, C_{\hat{P}_1}(H)] \not\leq F(G) \cap \hat{P}_{n-1} = C_{\hat{P}_{n-1}}(\hat{P}_n)$.
By the Proposition applied to the action of the group $\hat{P}_{n-1}FH$ on the module $\hat{P}_n/\Phi(\hat{P}_n)$ we get

$$Ker(C_{\hat{P}_{n-1}}(H) \text{ on } C_{\hat{P}_n/\Phi(\hat{P}_n)}(H)) = Ker(C_{\hat{P}_{n-1}}(H) \text{ on } \hat{P}_n/\Phi(\hat{P}_n)).$$

It follows now that Y does not centralize $C_{\hat{P}_n}(H)$ and hence $f(C_G(H) = n = f(G)$. This completes the proof. ■

THANK YOU !