4th Cemal Koch Algebra Days
Middle East Technical University, Ankara, Turkey
22-23 April 2016

Some Fascinating Features of Commutativity Degree in Finite Algebraic Structures

Karim Ahmadidelir

Department of Mathematics,
Tabriz Branch, Islamic Azad University,
Tabriz, Iran

In this short talk, we discuss some of fascinating and interesting features and aspects about commutativity degree in finite algebraic structures, such as semigroups, rings, groups and Moufang loops.

The study of commutativity degree of finite groups started more than half a century ago.

The study of commutativity degree of finite groups started more than half a century ago. The *commutativity degree* (or *commuting probability*) of a finite algebraic structure is defined to be the probability that two randomly chosen elements of that algebraic structure commute with each other.

The study of commutativity degree of finite groups started more than half a century ago. The *commutativity degree* (or *commuting probability*) of a finite algebraic structure is defined to be the probability that two randomly chosen elements of that algebraic structure commute with each other. In fact, one measures the abelianness (or commutativeness) of a finite algebraic structure A by counting the number of pairs of elements of A that commute.

Let us denote it by $Pr(A)$.

Let us denote it by $Pr(A)$. Formally, we have:

$$Pr(A) = \frac{|\{(x,y) \in A^2 \mid xy = yx\}|}{|A|^2} = \frac{\sum_{x \in A} |C_A(x)|}{|A|^2},$$

where $C_A(x)$ is the centralizer of x in A.

Let us denote it by $Pr(A)$. Formally, we have:

$$Pr(A) = \frac{\mid \{(x, y) \in A^2 \mid xy = yx\} \mid}{\mid A \mid^2} = \frac{\sum_{x \in A} \mid C_A(x) \mid}{\mid A \mid^2},$$

where $C_A(x)$ is the centralizer of x in A. For a finite group A it has been proved that $Pr(A) = \frac{k(A)}{|A|}$ where $k(A)$ is the number of conjugacy classes of A

Let us denote it by $Pr(A)$. Formally, we have:

$$Pr(A) = \frac{|\{(x,y) \in A^2 \mid xy = yx\}|}{|A|^2} = \frac{\sum_{x \in A} |C_A(x)|}{|A|^2},$$

where $C_A(x)$ is the centralizer of x in A. For a finite group A it has been proved that $Pr(A) = \frac{k(A)}{|A|}$ where $k(A)$ is the number of conjugacy classes of A (see [6, 8] for example).

The first surprising or fascinating known fact about $Pr(G)$, where G is a finite group, is that $Pr(G) \approx 1$ implies $Pr(G) = 1$.

The first surprising or fascinating known fact about $Pr(G)$, where G is a finite group, is that $Pr(G) \approx 1$ implies $Pr(G) = 1$. In other words, there is no finite group G with $\frac{5}{8} < Pr(G) < 1$.

The first surprising or fascinating known fact about $Pr(G)$, where G is a finite group, is that $Pr(G) \approx 1$ implies $Pr(G) = 1$. In other words, there is no finite group G with $\frac{5}{8} < Pr(G) < 1$. Also, it has been proved that every finite group G with $Pr(G) = \frac{5}{8}$ must be nilpotent (in fact, iff $\frac{G}{Z(G)} \cong C_2 \times C_2$), [6].

The first surprising or fascinating known fact about $Pr(G)$, where G is a finite group, is that $Pr(G) \approx 1$ implies $Pr(G) = 1$. In other words, there is no finite group G with $\frac{5}{8} < Pr(G) < 1$. Also, it has been proved that every finite group G with $Pr(G) = \frac{5}{8}$ must be nilpotent (in fact, iff $\frac{G}{Z(G)} \cong C_2 \times C_2$), [6]. There are other gaps in the set:

The first surprising or fascinating known fact about $Pr(G)$, where G is a finite group, is that $Pr(G) \approx 1$ implies $Pr(G) = 1$. In other words, there is no finite group G with $\frac{5}{8} < Pr(G) < 1$. Also, it has been proved that every finite group G with $Pr(G) = \frac{5}{8}$ must be nilpotent (in fact, iff $\frac{G}{Z(G)} \cong C_2 \times C_2$), [6]. There are other gaps in the set:

$$\mathfrak{P}_1 = \{Pr(G) \mid G \;\; is \;\; a \;\; finite \;\; group\}.$$

The first surprising or fascinating known fact about $Pr(G)$, where G is a finite group, is that $Pr(G) \approx 1$ implies $Pr(G) = 1$. In other words, there is no finite group G with $\frac{5}{8} < Pr(G) < 1$. Also, it has been proved that every finite group G with $Pr(G) = \frac{5}{8}$ must be nilpotent (in fact, iff $\frac{G}{Z(G)} \cong C_2 \times C_2$), [6]. There are other gaps in the set:

$$\mathfrak{P}_1 = \{Pr(G) \mid G \ \ is \ \ a \ \ finite \ \ group\}.$$

For example, there is no finite group G with $\frac{7}{16} < Pr(G) < \frac{1}{2}$;

The first surprising or fascinating known fact about $Pr(G)$, where G is a finite group, is that $Pr(G) \approx 1$ implies $Pr(G) = 1$. In other words, there is no finite group G with $\frac{5}{8} < Pr(G) < 1$. Also, it has been proved that every finite group G with $Pr(G) = \frac{5}{8}$ must be nilpotent (in fact, iff $\frac{G}{Z(G)} \cong C_2 \times C_2$), [6]. There are other gaps in the set:

$$\mathfrak{P}_1 = \{Pr(G) \mid G \ \ is \ \ a \ \ finite \ \ group\}.$$

For example, there is no finite group G with $\frac{7}{16} < Pr(G) < \frac{1}{2}$; however, there is a group G of the order 16 with $Pr(G) = \frac{7}{16}$ and $Pr(S_3) = \frac{1}{2}$, where S_3 is the symmetric group of degree 3.

So, a natural question occurs:

So, a natural question occurs:

"*What is the set* $\mathfrak{P}_1$ *look like?*"

So, a natural question occurs:

"*What is the set* $\mathfrak{P}_1$ *look like?*"

This question and some others first studied in general by K.S. Joseph in 1977, [7], who proposed the following three conjectures:

So, a natural question occurs:

"*What is the set* $\mathfrak{P}_1$ *look like?*"

This question and some others first studied in general by K.S. Joseph in 1977, [7], who proposed the following three conjectures:

(J_1) All limit points of $\mathfrak{P}_1$ are rational;

So, a natural question occurs:

"*What is the set* $\mathfrak{P}_1$ *look like?*"

This question and some others first studied in general by K.S. Joseph in 1977, [7], who proposed the following three conjectures:

(J_1) All limit points of $\mathfrak{P}_1$ are rational;

(J_2) $\mathfrak{P}_1$ is a well-ordered set by $>$;

So, a natural question occurs:

"*What is the set* $\mathfrak{P}_1$ *look like?*"

This question and some others first studied in general by K.S. Joseph in 1977, [7], who proposed the following three conjectures:

(J_1) All limit points of $\mathfrak{P}_1$ are rational;

(J_2) $\mathfrak{P}_1$ is a well-ordered set by $>$;

(J_3) $\mathfrak{P}_1 \cup \{0\}$ is a closed set.

Recently, S. Eberhard in [4], has shown that conjectures J_1 and J_2 are true, and so $\mathfrak{P}_1$ is nowhere dense.

Recently, S. Eberhard in [4], has shown that conjectures J_1 and J_2 are true, and so $\mathfrak{P}_1$ is nowhere dense. In this direction, he has used the so-called *Egyptian Fractions* and their properties.

Recently, S. Eberhard in [4], has shown that conjectures J_1 and J_2 are true, and so $\mathfrak{P}_1$ is nowhere dense. In this direction, he has used the so-called *Egyptian Fractions* and their properties.
Before him, Hegarty in 2013, has shown that conjectures J_1 and J_2 are true for the interval $(\frac{2}{9}, 1]$.

Recently, S. Eberhard in [4], has shown that conjectures J_1 and J_2 are true, and so $\mathfrak{P}_1$ is nowhere dense. In this direction, he has used the so-called *Egyptian Fractions* and their properties.
Before him, Hegarty in 2013, has shown that conjectures J_1 and J_2 are true for the interval $(\frac{2}{9}, 1]$.
But, Hegarty had used representation theory to prove his assertions.

Some years ago, the speaker of this talk has shown in [1] that inspite of groups, in finite semigroups, 1 is a limit point of the set:

Some years ago, the speaker of this talk has shown in [1] that inspite of groups, in finite semigroups, 1 is a limit point of the set:

$$\mathfrak{P}_2 = \{Pr(S) \mid S \ \ is \ \ a \ \ finite \ \ semigroup\}.$$

Some years ago, the speaker of this talk has shown in [1] that inspite of groups, in finite semigroups, 1 is a limit point of the set:

$$\mathfrak{P}_2 = \{Pr(S) \mid S \ \ is \ \ a \ \ finite \ \ semigroup\}.$$

He has presented an infinite class of finite non-commutative semigroups and proved that the commutativity degree of the semigroups in that class may be arbitrarily close to 1 and called this class of semigroups: *almost commutative* or *approximately abelian semigroups*.

Also, Givens (2008), Ponomarenko and Seilinski (2012) showed that $\mathfrak{P}_2$ is dense in $[0, 1]$ and $\mathfrak{P}_2 = \mathbb{Q} \cap [0, 1]$, respectively, Also, Givens (2008), Ponomarenko and Seilinski (2012) showed that $\mathfrak{P}_2$ is dense in $[0,1]$ and $\mathfrak{P}_2 = \mathbb{Q} \cap [0,1]$, respectively, and so the conjectures J_1 and J_2 are not true for finite semigroups.

Also, Givens (2008), Ponomarenko and Seilinski (2012) showed that $\mathfrak{P}_2$ is dense in $[0,1]$ and $\mathfrak{P}_2 = \mathbb{Q} \cap [0,1]$, respectively, and so the conjectures J_1 and J_2 are not true for finite semigroups. Therefore, $\mathfrak{P}_2 \cup \{0\}$ is not closed and J_3 is not also true for them.

Although, D. MacHale proved in 1976, [8], that there is no finite ring R with $\frac{5}{8} < Pr(R) < 1$;

Although, D. MacHale proved in 1976, [8], that there is no finite ring R with $\frac{5}{8} < Pr(R) < 1$; however, there is a ring R of the order 8 with $Pr(R) = \frac{5}{8}$, and so the bound $\frac{5}{8}$ is the best possible.

Although, D. MacHale proved in 1976, [8], that there is no finite ring R with $\frac{5}{8} < Pr(R) < 1$; however, there is a ring R of the order 8 with $Pr(R) = \frac{5}{8}$, and so the bound $\frac{5}{8}$ is the best possible. But for the time being, we do not know anything about the Joseph's conjectures in finite rings.

A set Q with one binary operation is a quasigroup if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified.

A set Q with one binary operation is a quasigroup if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified.

Loop is a quasigroup with a neutral element 1 satisfying $1x = x1 = x$ for every x.

A set Q with one binary operation is a quasigroup if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified.

Loop is a quasigroup with a neutral element 1 satisfying $1x = x1 = x$ for every x.

Moufang loops are loops in which any of the (equivalent) Moufang identities:

A set Q with one binary operation is a quasigroup if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified.

Loop is a quasigroup with a neutral element 1 satisfying $1x = x1 = x$ for every x.

Moufang loops are loops in which any of the (equivalent) Moufang identities:

$$
\begin{aligned}
((xy)x)z &= x(y(xz)), && (M_1)\\
x(y(zy)) &= ((xy)z)y, && (M_2)\\
(xy)(zx) &= x((yz)x), && (M_3)\\
(xy)(zx) &= (x(yz))x. && (M_4)
\end{aligned}
$$

holds.

The speaker has conjectured that just like groups, in finite Moufang loops, there is no finite Moufang loop M with $\frac{23}{32} < Pr(M) < 1$.

The speaker has conjectured that just like groups, in finite Moufang loops, there is no finite Moufang loop M with $\frac{23}{32} < Pr(M) < 1$. Actually, for an important class of finite Moufang loops called Chain loops and its modifications, the same facts that are satisfied by groups are also true, [2].

The speaker has conjectured that just like groups, in finite Moufang loops, there is no finite Moufang loop M with $\frac{23}{32} < Pr(M) < 1$. Actually, for an important class of finite Moufang loops called Chain loops and its modifications, the same facts that are satisfied by groups are also true, [2]. Specially, the analogous of conjectures J_1 and J_2 are true for the class of finite Chain loops and its modifications.

The speaker has conjectured that just like groups, in finite Moufang loops, there is no finite Moufang loop M with $\frac{23}{32} < Pr(M) < 1$. Actually, for an important class of finite Moufang loops called Chain loops and its modifications, the same facts that are satisfied by groups are also true, [2]. Specially, the analogous of conjectures J_1 and J_2 are true for the class of finite Chain loops and its modifications. So, the set:

$$\mathfrak{P}_3 = \{Pr(M) \mid M \ \ is \ \ a \ \ finite \ \ Chain \ \ loop\}$$

The speaker has conjectured that just like groups, in finite Moufang loops, there is no finite Moufang loop M with $\frac{23}{32} < Pr(M) < 1$. Actually, for an important class of finite Moufang loops called Chain loops and its modifications, the same facts that are satisfied by groups are also true, [2]. Specially, the analogous of conjectures J_1 and J_2 are true for the class of finite Chain loops and its modifications. So, the set:

$$\mathfrak{P}_3 = \{Pr(M) \mid M \ \ is \ \ a \ \ finite \ \ Chain \ \ loop\}$$

is nowhere dense and well-ordered by $>$.

Now, since by J_2, $\mathfrak{P}_1$ is a well-ordered set, another question is:

Now, since by J_2, $\mathfrak{P}_1$ is a well-ordered set, another question is:

"What is the order type of $\mathfrak{P}_1$?"

Now, since by J_2, $\mathfrak{P}_1$ is a well-ordered set, another question is:

"What is the order type of $\mathfrak{P}_1$?"

Eberhard has proved in [4] that the order type of $(\mathfrak{P}_1, >)$ is either ω^ω or ω^{ω^2} (only two posibilities).

Now, since by J_2, $\mathfrak{P}_1$ is a well-ordered set, another question is:

"What is the order type of $\mathfrak{P}_1$?"

Eberhard has proved in [4] that the order type of $(\mathfrak{P}_1, >)$ is either ω^ω or ω^{ω^2} (only two posibilities). So, we deduce that the order type of $(\mathfrak{P}_3, >)$ is also either ω^ω or ω^{ω^2}.

Now, since by J_2, $\mathfrak{P}_1$ is a well-ordered set, another question is:

"*What is the order type of* $\mathfrak{P}_1$*?*"

Eberhard has proved in [4] that the order type of $(\mathfrak{P}_1, >)$ is either ω^{ω} or ω^{ω^2} (only two posibilities). So, we deduce that the order type of $(\mathfrak{P}_3, >)$ is also either ω^{ω} or ω^{ω^2}. As the same way, we may ask the similar questions for finite rings, semigroups and Moufang loops.

Also recently, the speaker has defined a new notion, the *associativity degree* (or *associating probability*) of a finite loop L, denoted by $Pas(L)$, as the probability that three (randomly chosen) elements of L associate with respect to its operation.

Also recently, the speaker has defined a new notion, the *associativity degree* (or *associating probability*) of a finite loop L, denoted by $Pas(L)$, as the probability that three (randomly chosen) elements of L associate with respect to its operation. Formally, we have:

Also recently, the speaker has defined a new notion, the *associativity degree* (or *associating probability*) of a finite loop L, denoted by $Pas(L)$, as the probability that three (randomly chosen) elements of L associate with respect to its operation. Formally, we have:

$$Pas(L) = \frac{|\{(x, y, z) \in L^3 \mid x(yz) = (xy)z\}|}{|L^3|}.$$

Also recently, the speaker has defined a new notion, the *associativity degree* (or *associating probability*) of a finite loop L, denoted by $Pas(L)$, as the probability that three (randomly chosen) elements of L associate with respect to its operation. Formally, we have:

$$Pas(L) = \frac{|\{(x, y, z) \in L^3 \mid x(yz) = (xy)z\}|}{|L^3|}.$$

Then he has tried to obtain a best upper bound for $Pas(M)$, where M is a finite non-associative *Moufang loop*.

Also recently, the speaker has defined a new notion, the *associativity degree* (or *associating probability*) of a finite loop L, denoted by $Pas(L)$, as the probability that three (randomly chosen) elements of L associate with respect to its operation. Formally, we have:

$$Pas(L) = \frac{|\{(x, y, z) \in L^3 \mid x(yz) = (xy)z\}|}{|L^3|}.$$

Then he has tried to obtain a best upper bound for $Pas(M)$, where M is a finite non-associative *Moufang loop*. He has still shown that for the class of Chain loops, and its modifications, this best upper bound is $\frac{43}{64}$ and it is related to the commutativity degree of M, $Pr(M)$.

Also recently, the speaker has defined a new notion, the *associativity degree* (or *associating probability*) of a finite loop L, denoted by $Pas(L)$, as the probability that three (randomly chosen) elements of L associate with respect to its operation. Formally, we have:

$$Pas(L) = \frac{|\{(x, y, z) \in L^3 \mid x(yz) = (xy)z\}|}{|L^3|}.$$

Then he has tried to obtain a best upper bound for $Pas(M)$, where M is a finite non-associative *Moufang loop*. He has still shown that for the class of Chain loops, and its modifications, this best upper bound is $\frac{43}{64}$ and it is related to the commutativity degree of M, $Pr(M)$. Here is also, the conjecture is: *for any finite Moufang loop M, $Pas(M) \leq \frac{43}{64}$*, [3].

We know that for the commutativity degree, $Pr(L) = 1$ iff L is commutative.

We know that for the commutativity degree, $Pr(L) = 1$ iff L is commutative. Here is also, $Pas(L) = 1$ iff L is associative (so is a group).

We know that for the commutativity degree, $Pr(L) = 1$ iff L is commutative. Here is also, $Pas(L) = 1$ iff L is associative (so is a group). Therefore, by the above facts we deduce that the set:

We know that for the commutativity degree, $Pr(L) = 1$ iff L is commutative. Here is also, $Pas(L) = 1$ iff L is associative (so is a group). Therefore, by the above facts we deduce that the set:

$$\mathfrak{P}_4 = \{Pas(M) \mid M \;\; is \;\; a \;\; finite \;\; Chain \;\; loop\}$$

We know that for the commutativity degree, $Pr(L) = 1$ iff L is commutative. Here is also, $Pas(L) = 1$ iff L is associative (so is a group). Therefore, by the above facts we deduce that the set:

$$\mathfrak{P}_4 = \{Pas(M) \mid M \ \ is \ \ a \ \ finite \ \ Chain \ \ loop\}$$

is nowhere dense and well-ordered by $>$, and also, the order type of $(\mathfrak{P}_4, >)$ is either ω^ω or ω^{ω^2}.

Finally, some problems those have been answered about commutativity degree in finite groups arise about Moufang loops too, like as:

Finally, some problems those have been answered about commutativity degree in finite groups arise about Moufang loops too, like as:
More generally, it has been proved in group theory that for a finite group G, if $Pr(G) > \frac{1}{12}$, $Pr(G) > \frac{1}{3}$ and $Pr(G) > \frac{1}{2}$, then G is solvable, supersolvable and nilpotent, respectively.

Finally, some problems those have been answered about commutativity degree in finite groups arise about Moufang loops too, like as:
More generally, it has been proved in group theory that for a finite group G, if $Pr(G) > \frac{1}{12}$, $Pr(G) > \frac{1}{3}$ and $Pr(G) > \frac{1}{2}$, then G is solvable, supersolvable and nilpotent, respectively. So, $Pr(G) = \frac{1}{12}$, $\frac{1}{3}$ and $\frac{1}{2}$ are the least upper bound on the commutativity degree of non–solvable, non–supersolvable and non–nilpotent groups, respectively.

Finally, some problems those have been answered about commutativity degree in finite groups arise about Moufang loops too, like as:
More generally, it has been proved in group theory that for a finite group G, if $Pr(G) > \frac{1}{12}$, $Pr(G) > \frac{1}{3}$ and $Pr(G) > \frac{1}{2}$, then G is solvable, supersolvable and nilpotent, respectively. So, $Pr(G) = \frac{1}{12}$, $\frac{1}{3}$ and $\frac{1}{2}$ are the least upper bound on the commutativity degree of non–solvable, non–supersolvable and non–nilpotent groups, respectively.

Question 1: *Can we determine the structure of a given finite Moufang loop by its commutativity and/or associativity degrees (such as nilpotency, solvability, simplicity and so on)?*

If G is a non–abelian finite simple group, then $Pr(G) \leq 1/12$, with equality for the alternating group of degree 5, A_5.

If G is a non–abelian finite simple group, then $Pr(G) \leq 1/12$, with equality for the alternating group of degree 5, A_5.

Question 2: *Is there a similar upper bound for a non–abelian finite simple Moufang loop?*

If G is a non–abelian finite simple group, then $Pr(G) \leq 1/12$, with equality for the alternating group of degree 5, A_5.

Question 2: *Is there a similar upper bound for a non–abelian finite simple Moufang loop?* (For example, the commutativty and associativity degrees of Paige loop of order 120, which is simple, is $4/25$ and $13/125$, respectively.)

It is well-known that there is no finite group G such that $\frac{7}{16} < Pr(G) < \frac{1}{2}$ (see [6] or [8]).

It is well-known that there is no finite group G such that $\frac{7}{16} < Pr(G) < \frac{1}{2}$ (see [6] or [8]). So, by Theorem main theorem, there is no finite non–associative Chain loop $M = M(G, 2)$ such that $\frac{65}{128} < Pas(M) < \frac{9}{6}$.

It is well-known that there is no finite group G such that $\frac{7}{16} < Pr(G) < \frac{1}{2}$ (see [6] or [8]). So, by Theorem main theorem, there is no finite non–associative Chain loop $M = M(G, 2)$ such that $\frac{65}{128} < Pas(M) < \frac{9}{6}$.

Question 3: *Can we extend this result for all finite Moufang loops?*

It is well-known that there is no finite group G such that $\frac{7}{16} < Pr(G) < \frac{1}{2}$ (see [6] or [8]). So, by Theorem main theorem, there is no finite non–associative Chain loop $M = M(G, 2)$ such that $\frac{65}{128} < Pas(M) < \frac{9}{6}$.

Question 3: *Can we extend this result for all finite Moufang loops?*

By computations with the aim of GAP [5], for all of the non–associative Moufang loops of order n upto 64, $n = 81$ and $n = 243$, the associativity and the commutativity degrees are not equal.

It is well-known that there is no finite group G such that $\frac{7}{16} < Pr(G) < \frac{1}{2}$ (see [6] or [8]). So, by Theorem main theorem, there is no finite non–associative Chain loop $M = M(G, 2)$ such that $\frac{65}{128} < Pas(M) < \frac{9}{6}$.

Question 3: *Can we extend this result for all finite Moufang loops?*

By computations with the aim of GAP [5], for all of the non–associative Moufang loops of order n upto 64, $n = 81$ and $n = 243$, the associativity and the commutativity degrees are not equal.

Question 4: *Is there a finite non–associative Moufang loop M, with $Pas(M) = Pr(M)$?*

K. Ahmadidelir, C.M. Campbell, and H. Doostie, *Two classes of finite semigroups and monoids involving Lucas numbers*, Semigroup Forum, **78** (2009), 200–209.

K. Ahmadidelir, *On the Commutativity Degree in Finite Moufang Loops*, to appear in *Int. J. Group Theory (IJGT)*, Articles in Press, Accepted Manuscript, Available Online from 02 March 2015.

K. Ahmadidelir, *On the Associativity Degree in Finite Moufang Loops*, submitted to Int. J. Alg. & Comp., (2015).

S. Eberhard, *Commuting probabilities of finite groups*, Bull. Lond. Math. Soc., **47**(5) (2015), 796–808.

The GAP group, *GAP - Groups, Algorithms and Programming*, Aachen, St. Andrews **Version 4.7.2** (2013), (http://www.gap–system.org).

W.H. Gustafson, *What is the probability that two group elements commute?*, Amer. Math. Monthly, **80** (1973), 1031–1034.

K.S. Joseph, *Several Conjectures on Commutativity in Aalgebraic structures*, Amer. Math. Monthly, **84** (1977), 550–551.

D. MacHale, *Commutativity in finite rings*, Amer. Math. Monthly, **83** (1976), 30–32.

Many Thanks for Your Attention and Patience

Many Thanks for Your Attention and Patience