Kaplansky zero divisor conjecture on group algebras over torsion-free groups

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What is a Group Ring?

- Donald S. Passman says in the Preface of his book: *Infinite Group Rings*, Marcel Dekker, New York, 1971:

Figure : Donald S. Passman (1940-)

- The group ring $K[G]$ is an associative ring which exhibits properties of the group G and the field of coefficients K. As the name implies, its study is a meeting place for group theory and ring theory, and as such it has been approached from many different points of view. For example, the finite group theorist does character theory and the analyst does operator theory and Fourier analysis.

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- This algebraic study of group rings was initiated in 1949 by I. Kaplansky, but it did not really catch on until the fundamental work of S. A. Amitsur appeared some ten years later. Since then the subject has been pursued by a small but growing number of researchers, and at this point in time it has reached the stage in its development where a coherent account of the basic results is needed.

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Figure : Irving Kaplansky (1917-2006)

Figure : Shimshon Avraham Amitsur (1921-1994)

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- $\sum_{x \in G} \alpha_x x \cdot \sum_{x \in G} \beta_x x = \sum_{x \in G} \left(\sum_{y \in G} \alpha_y \beta_{y^{-1}x} \right) x$.

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- If we assume R has the identity, then $R[G]$ has an identity. This is $1_R 1_G$ which is the function from G to R mapping all elements $x \in G \setminus \{1_G\}$ to 0_R and 1_G to 1_R.

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- Let a be an element of finite order $n > 1$ in a group. Then $(a-1)(a^{n-1} + \cdots + a + 1) = a^n - 1 = 0$ in the group ring RG for any ring R with identity. This means that a necessary condition to not have zero divisors in a group ring is the torsion-freeness of the group.

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- **(Kaplansky Zero Divisor Conjecture (KZDC))**. For a torsion-free group G and an integral domain R, the group ring RG contains no zero divisor.

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- KZDC is valid for u.p.-groups.

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- (P. H. Kropholler, P. A. Linnell and J. A. Moody, 1988) KZDC is valid for torsion-free elementary amenable groups.
- The class of elementary amenable groups contains all solvable and finite groups and it is closed under taking subgroup and extensions.

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- If $KZDC(\mathbb{F}, G)$ is non-empty, it contains a minimum in $\mathbb{N} \times \mathbb{N}$ with respect to the lexicographic order $<$,i.e., $(n, m) < (n', m')$ if and only if $n < n'$ or $n = n'$ and $m < m'$.

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- The set $KZDC(\mathbb{F}, G)$ is symmetric, i.e., $(n, m) \in KZDC(\mathbb{F}, G) \Leftrightarrow (m, n) \in KZDC(\mathbb{F}, G)$. This is because

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- If R is a commutative ring, for any group ring RG, the map $i : \sum_{x\in G} r_x x \mapsto \sum_{x\in G} r_x x^{-1}$ is an anti ring automorphism such that i^2 is the identity map RG.

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- Thus if $\alpha, \beta \in RG$ such that $\alpha\beta = 0$ then $i(\beta)i(\alpha) = 0$. Since $\mathrm{Supp}(\alpha)^{-1} = \mathrm{Supp}(i(\alpha))$ for any element α of a group ring, it follows that the set $KZDC(\mathbb{F}, G)$ is symmetric.

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- Let $\mathbb{F}_p$ be the field of prime size p. If $(n, m) \in KZDC(\mathbb{Q}, G)$ then $(n', m') \in KZDC(\mathbb{F}_p, G)$ for some $n' \leq n$ and $m' \leq m$.

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- (Pascal Schweitzer 2013) For any torsion-free group G, if $(n, m) \in KZDC(\mathbb{Q}, G)$ then
  (i) $n > 2$, (ii) $m > 2$,
  (iii) $n > 3$ or $m > 16$, (iv) $n > 16$ or $m > 3$,
  (v) $n > 4$ or $m > 7$, (vi) $n > 7$ or $m > 4$.

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- Pascal Schweitzer says that "It is thus sufficient to check the conjecture only for length combinations for which $\text{length}(\alpha) = \text{length}(\beta)$. However, in the construction that, given a zero divisor produces an element of square zero, it is not clear how the length changes."

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- (A.) If $\alpha, \beta \in \mathbb{F}G \setminus \{0\}$ such that $\alpha\beta = 0$, then $\gamma = \beta x \alpha \neq 0$ for some $x \in G$. Since $\text{length}(\beta x \alpha) \leq \text{length}(\beta)\text{length}(x\alpha) = \text{length}(\alpha)\text{length}(\beta)$, it follows that if $(\ell,\ell) \notin KZDC(\mathbb{F}, G)$ for all $\ell \leq n$ for some n, then $(s,t) \notin KZDC(\mathbb{F}, G)$ for all positive integers s, t such that $st \leq n$. It is because $\gamma^2 = 0$ and $\gamma \neq 0$.

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- Suppose that $\beta x \alpha = 0$ for all $x \in G$. Then $\theta(\beta')\theta(\alpha') = 0$, where $\beta' = b^{-1}\beta$ and $\alpha' = \alpha a^{-1}$ and $b \in \mathrm{Supp}(\beta)$ and $a \in \mathrm{Supp}(\alpha)$ are arbitrary elements. Note that $1_G \in \mathrm{Supp}(\alpha') \cap \mathrm{Supp}(\beta')$ and so $\theta(\beta') \neq 0$ and $\theta(\alpha') \neq 0$. This is not possible, as the supports of $\theta(\beta')$ and $\theta(\alpha')$ generate a torsion-free abelian group.

Possible Zero Divisors of Length 3 in $\mathbb{Z}G$

- Let G be a torsion-free group and let $\alpha = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3 \in \mathbb{Z}G$ and length$(\alpha) = 3$. Suppose that α is a zero divisor.

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- We may assume that gcd$(\alpha_1, \alpha_2, \alpha_3) = 1$. If $|\alpha_i| > 1$ for some i, then p divides α_i for some prime p. It follows that the image of α in $\mathbb{F}_p G$ is a zero divisor of length at most 2, which is not possible.

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- Therefore $\{\alpha_1, \alpha_2, \alpha_3\} \subseteq \{-1, 1\}$ and so a possible zero divisor of length 3 in $\mathbb{Z}G$ has one of the following forms: $nh_1 + nh_2 + nh_3$ or $\pm(nh_1 - nh_2 + nh_3)$ for some non-zero integer n.

A Graph Arising from the Existence of Zero Divisors of Length 3 in $\mathbb{F}_2 G$

- Let $h_1 + h_2 + h_3$ be a possible zero divisor in $F_2 G$ such that $(h_1 + h_2 + h_3)(g_1 + \cdots + g_n) = 0$, where $g_1 + \cdots + g_n \in \mathbb{F}_2 G$ of length n.

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- Consider the graph Γ whose vertex set is $\{g_1, \ldots, g_n\}$ and two distinct vertices g_i and g_j are adjacent if $h_\ell g_i = h_k g_j$ for some $\ell, k \in \{1, 2, 3\}$.

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- The graph Γ is the induced subgraph of the Cayley graph of the subgroup $H = \langle h_1, h_2, h_3 \rangle$ with respect to the set $S = \{h_\ell^{-1} h_k \mid \ell \neq k, \ell, k \in \{1, 2, 3\}\}$.

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