# A SEMI-HISTORICAL INTRODUCTION TO LEAVITT PATH ALGEBRAS

Murad Özaydin
Department of Mathematics
University of Oklahoma, Norman, OK, USA
E-mail: mozaydin@ou.edu

LPAs (Leavitt Path Algebras) were defined just over a decade ago (Abrams and Aranda Pino, 2005; Ara, Moreno and Pardo, 2007) but they have roots in the works of Leavitt in the 60s focused on understanding the extent of the failure of the IBN (Invariant Basis Number) property for arbitrary rings. A ring has IBN if any two bases of a finitely generated free module have the same number of elements. Fields, division rings, commutative rings, Noetherian rings all have IBN. A classical example of a ring without IBN is the algebra of endomorphisms of a countably infinite dimensional vector space. The free module of rank 1 over this ring has bases of $n$ elements for any positive integer $n$. In the early 60s Bill Leavitt asked and then answered this question: Given any $m < n$ (positive integers) is there a ring $R$ having a free module with a basis of $m$ elements and another basis with $n$ elements but no bases with $k$ elements if $k < n$ and not equal to $m$?

The algebras Leavitt constructed are now called the Leavitt algebras and denoted by $L(m, n)$. They are simple iff $m = 1$ and have a semi-universal property among the algebras whose free module of rank $m$ also has a basis of $n$ elements. That a free $L(m, n)$ module of rank m has a basis of $n$ elements is immediate from the definition. However, to show that there are no bases of $k$ elements ($k < n$, different from $m$) is essentially a question of nonstable K-theory and highly nontrivial. This was put in a broader context by George Bergman in the 70s by constructing rings $R$ with essentially arbitrary $V(R)$ (= the monoid of isomorphism classes of finitely generated projective $R$-modules under direct sum).

$L(1, n)$ is also a Cohn localization (defined by P. M. Cohn in the 70s) of the noncommutative polynomial algebra in $n$ variables. Since this algebra is the path (or quiver) algebra of the rose with n petals, with 20/20 hindsight we see a glimmer of the connection with path algebras. In fact it took three more decades and a detour through Functional Analysis for the LPAs to be defined.

The $C^*$-algebras defined by Joachim Cuntz (1977) and later generalizations by Cuntz-Krieger, Pimsner and others led to the theory of graph $C^*$-algebras developed in the 90s (and still active). Now the algebra is defined by a di(rected )graph so the combinatorial properties of the graph yield corresponding algebraic properties (both for graph $C^*$-algebras and the LPAs). LPAs include matrix algebras, the Jacobson-Toeplitz algebra, quantum spheres and many others as well as the Leavitt algebras. Most of these algebras have IBN! The closure of an LPA with respect to an appropriate norm yields the corresponding $C^*$-graph algebra.

The rings $L(1, n)$ defined by Leavitt and their analytic cousins, the $C^*$-algebras of Cuntz are not artificial or pathological structures constructed only for the sake of providing counterexamples; for instance they implicitly come up in Signal Processing (as the algebras generated by the downsampling and upsampling operators). Moreover Leavitt's work provided important impetus for major developments in noncommutative ring theory in the 70s by Cohn, Bergman and others.

I plan to start with the basic definitions, state some fundamental results, explain the criterion for an LPA to have IBN (joint work with Muge Kanuni Er) and, if time permits, indicate the ideas involved in the recent classification of the finite dimensional representations (jointly with Ayten Koc). While LPAs are (Cohn) localizations of Path (or Quiver) Algebras whose finite dimensional representations are usually wild, the category of finite dimensional representations of LPAs turn out to be tame with a very reasonable classification of all the indecomposables and the simples. All finite dimensional quotients of LPAs are also easy to describe.