Finite groups admitting a dihedral group of automorphisms

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Let F be a nilpotent group acted on by a group H via automorphisms and let the group G admit the semidirect product FH as a group of automorphisms so that $C_G(F) = 1$. By a well known result [1] due to Belyaev and Hartley, the solvability of G is consequence of the fixed point free action of the nilpotent group F. A lot of research, [7],[10], [11], [13], [14], [15] investigating the structure of G has been conducted in case where FH is a Frobenius group with kernel F and complement H. So the immediate question one could ask was whether the condition of being Frobenius for FH could be weakened or not. In this direction we introduced the concept of a Frobenius-like group in [8] as a generalization of Frobenius group and investigated the structure of G when the group FH is Frobenius-like [3],[4],[5],[6]. In particular, we obtained in [3] the same conclusion as in [10]; namely the nilpotent lengths of G and $C_G(H)$ are the same, when the Frobenius group FH is replaced by a Frobenius-like group under some additional assumptions. In a similar attempt in [16] Shumyatsky considered the case where FH is a dihedral group and proved the following:

Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β and let $F = \langle \alpha\beta \rangle$. (Here, $D = FH$ where $H = \langle \alpha \rangle$) Suppose that D acts on the group G by automorphisms in such a way that $C_G(F) = 1$. If $C_G(\alpha)$ and $C_G(\beta)$ are both nilpotent then G is nilpotent.

In the present paper we extend his result as follows:

Theorem. *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β and let $F = \langle \alpha\beta \rangle$. Suppose that D acts on the group G by automorphisms in such a way that $C_G(F) = 1$. Then the nilpotent length of G is equal to the maximum of the nilpotent lengths of the subgroups $C_G(\alpha)$ and $C_G(\beta)$.*

After completing the proof we became aware of the paper [2] by de Melo and seen that the above theorem follows as a corollary of the main theorem of the paper of de Melo, which is given below:

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Theorem. *Let $M = FH$ be a finite group that is a product of a normal abelian subgroup F and an abelian subgroup H. Assume that all elements in $M \backslash F$ have prime order p, and F has at most one subgroup of order p. Suppose that M acts on a finite group G in such a manner that $C_G(F) = 1$. Then $F_i(C_G(H)) = F_i(G) \cap C_g(H)$ for all i and $h(G) \leq h(C_G(H)) + 1$.*

The proof we give relies on the investigation of D-towers in G in the sense of [17] and the following proposition which, we think, can be effectively used in similar situations.

Proposition. *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β. Suppose that D acts on a q-group Q for some prime q and let V be a kQD-module for a field k of characteristic different from q such that the group $F = \langle \alpha\beta \rangle$ acts fixed point freely on the semidirect product VQ. If $C_Q(\alpha)$ acts nontrivially on V then we have $C_V(\alpha) \neq 0$ and $Ker(C_Q(\alpha)\, on\, C_V(\alpha)) = Ker(C_Q(\alpha)\, on\, V)$.*

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