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KAPLANSKY ZERO DIVISOR CONJECTURE ON GROUP ALGEBRAS OVER TORSION-FREE GROUPS

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Abstract. Let G be a any torsion-free group and R be an arbitrary commutative integral domain. Kaplansky's Zero Divisor conjecture states that the group ring $R[G]$ has no zero divisor, that is if $ab = 0$ for some $a, b \in R[G]$, then $a = 0$ or $b = 0$. It is known that $R[G]$ has no zero divisor with support of size at most 2. We will talk about the possible zero divisors in $R[G]$ whose supports have size 3, where R is the field $\mathbb{F}_2$ of order 2 or the ring of integers $\mathbb{Z}$. In particular we prove that if a zero divisor with support of size 3 exists in $\mathbb{Z}[G]$, then there exists a zero divisor in $\mathbb{Z}[G]$ whose support is contained in $\{-1, 1\}$.

1. **Introduction and Results**

A non-zero element α of a ring R is called a zero divisor if $\alpha\beta = 0$ or $\beta\alpha = 0$ for some non-zero element $\beta \in R$. A ring R is called a domain if R has no zero divisor.

Irving Kaplansky proposed the following famous question about the zero divisors of group algebras over torsion-free groups:

Question 1.1 (Problem 6 of [6])**.** Let G be an arbitrary torsion-free group and $\mathbb{F}$ be any field. Is it true that the group algebra $\mathbb{F}[G]$ a domain?

Question 1.1 is mostly known as Kaplansky Zero Divisor Conjecture. This is known to be true for any field $\mathbb{F}$ and one-sided orderable groups G [8]; for amalgamated free products G when the group ring of the subgroup over which the amalgam is formed satisfies the Ore condition [10]; supersolvable groups [5]; polycyclic-by-finite groups (see [1], [4] and [13]); elementary amenable groups [7]; one-relator groups [11]; congruence subgroups [9] and [3]; and certain hyperbolic groups [2].

Zero divisors with small support in group rings of torsion-free groups have been studied in [12]. It is fairly easy to show that $R[G]$ has no zero divisor with support of size at most 2.

Here we study zero divisors whose support are of size 3. Some of our results are the following:

Theorem 1.2. *Let G be an arbitrary torsion-free group. If a zero divisor with support of size 3 exists in $\mathbb{Z}[G]$, then there exists a zero divisor in $\mathbb{Z}[G]$ whose support is contained in $\{-1, 1\}$.*

A simple graph can be associated to a possible zero divisor with support of size 3 in the group algebra $\mathbb{F}_2[G]$ for a possible torsion-free group G. The graph is

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introduced in [12]. In [12] it is proved that the graph cannot have a triangle. We have proved that the graph cannot contain more than 20 other subgraphs.

Some forbidden subgraphs of the zero divisor graph

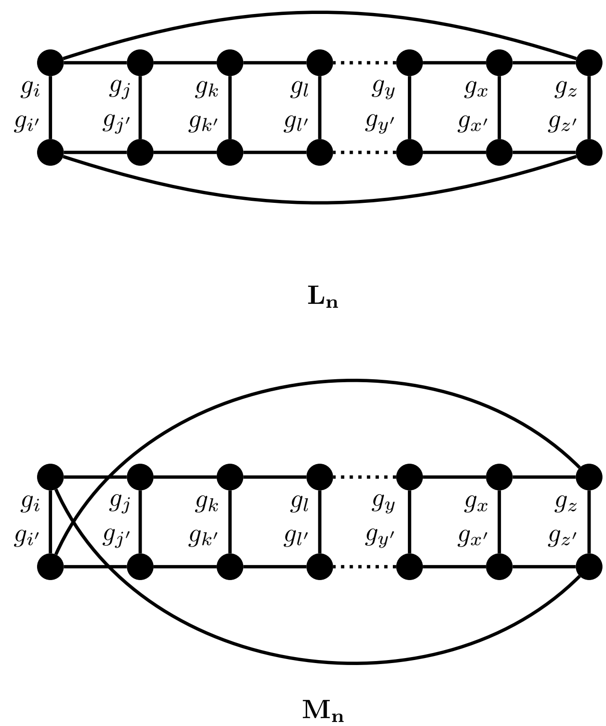


FIGURE 1. Two graphs which are not isomorphic to the zero divisor graph Γ

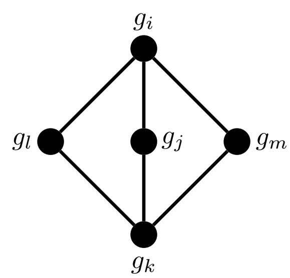


FIGURE 2. The complete bipartite graph $K_{2,3}$, a forbidden subgraph of the zero divisor graph

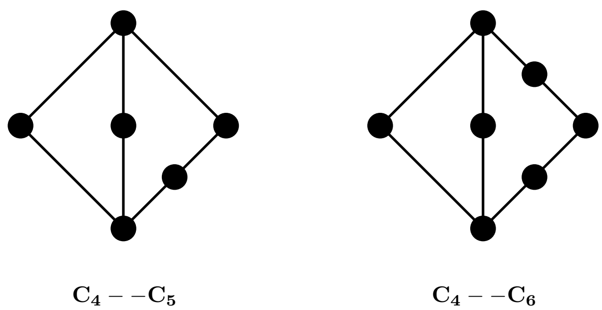


FIGURE 3. $(C_4 - -C_5)$ and $(C_4 - -C_6)$, two forbidden subgraphs of the zero divisor graph Γ

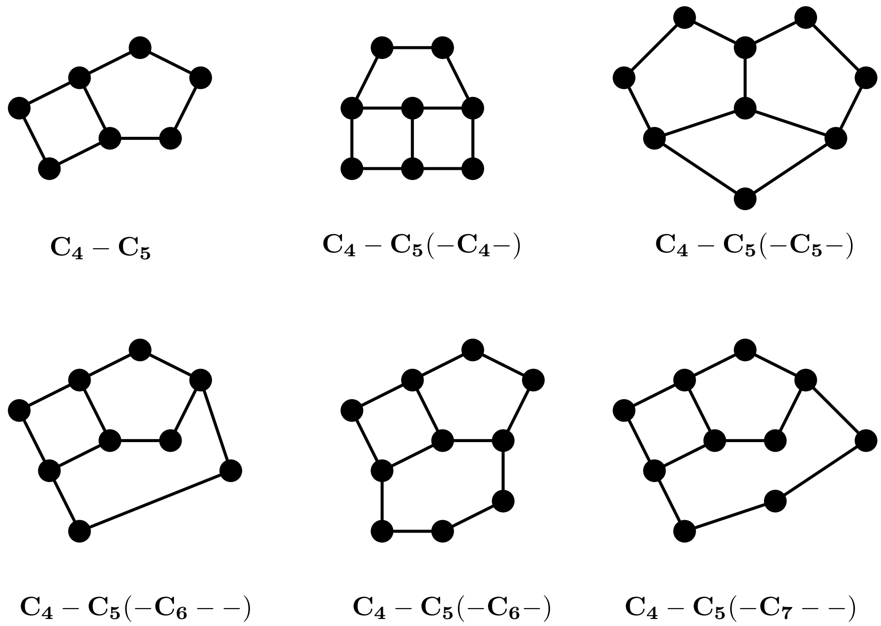
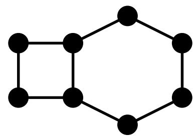
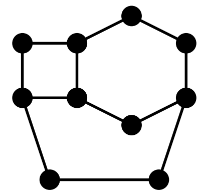


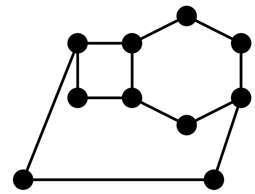
FIGURE 4. The graph $C_4 - C_5$ and some forbidden subgraphs which contains this graph



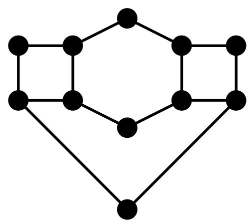
$\mathbf{C_4 - C_6}$



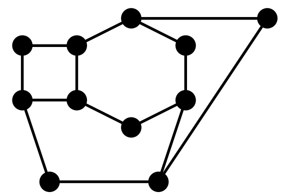
$\mathbf{C_4 - C_6}(-\mathbf{C_6}--)$



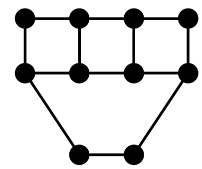
$\mathbf{C_4 - C_6}(--\mathbf{C_7}--)$



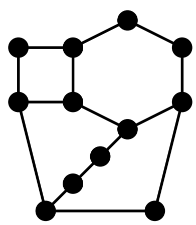
$\mathbf{C_4 - C_6}(-\mathbf{C_6}--)(-\mathbf{C_4}-)$



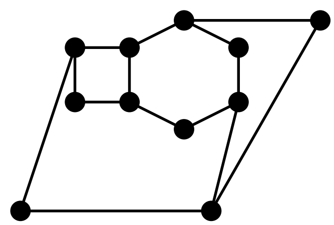
$\mathbf{C_4 - C_6}(-\mathbf{C_6}--)(--\mathbf{C_5}-)$



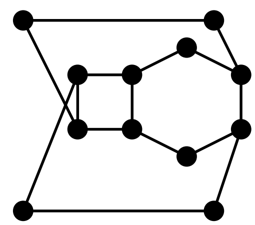
$\mathbf{C_4 - C_6}(-\mathbf{C_6}--)(---\mathbf{C_4})$



$\mathbf{C_4 - C_6}(-\mathbf{C_6}--)(\mathbf{C_6}---)$

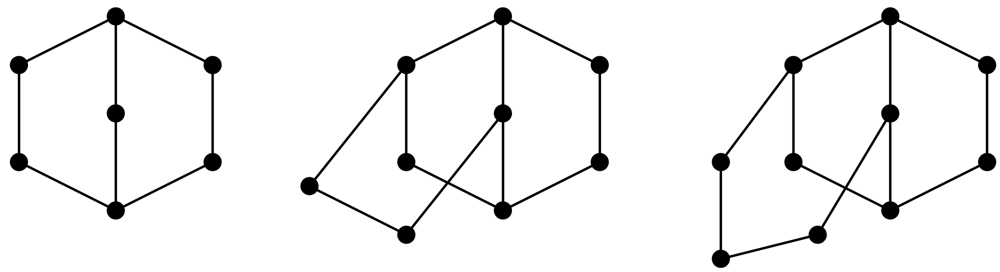


$\mathbf{C_4 - C_6}(--\mathbf{C_7}--)(--\mathbf{C_5}-)$



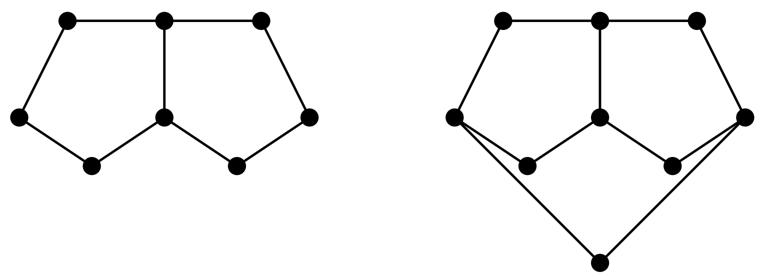
$\mathbf{C_4 - C_6}(--\mathbf{C_7}--)(--\mathbf{C_7}--)$

FIGURE 5. The graphs C_4-C_6, $C_4-C_6(-C_6--)$ and $C_4-C_6(--C_7--)$, and some forbidden subgraphs which contains one of them



$\mathbf{C_5 - -C_5}$ $\mathbf{C_5 - -C_5}(- - \mathbf{C_5})$ $\mathbf{C_5 - -C_5}(- - \mathbf{C_6})$

FIGURE 6. The graph $C_5 - -C_5$ and some forbidden subgraphs which contains it



$\mathbf{C_5 - C_5}$ $\mathbf{C_5 - C_5}(- - \mathbf{C_6} - -)$

FIGURE 7. The graph $C_5 - C_5$ and a forbidden subgraph which contains it

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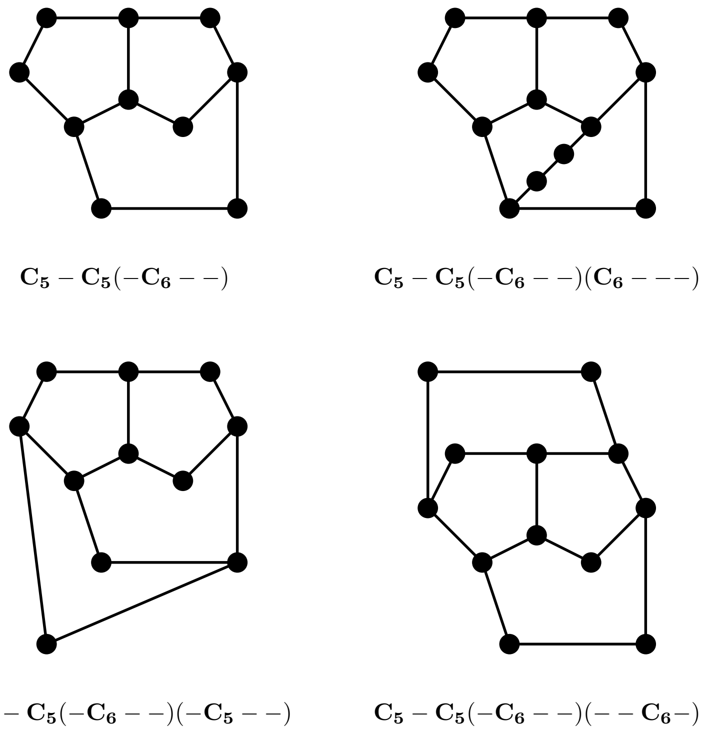

FIGURE 8. The graph $C_5 - C_5(-C_6 --)$ and some forbidden subgraphs which contains it

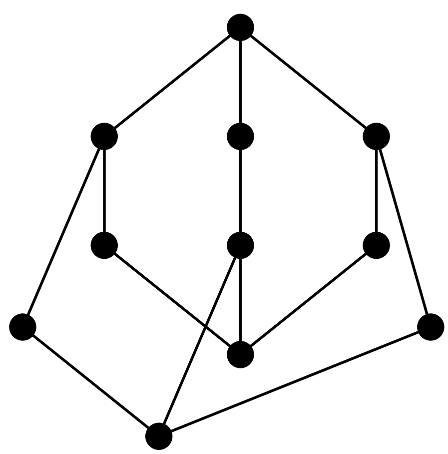

FIGURE 9. The forbidden subgraph $C6 --- C6(C6 --- C6)$

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Some Fascinating Features of Commutativity Degree in Finite Algebraic Structures

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In this short talk, we discuss some of fascinating and interesting features and aspects about commutativity degree in finite algebraic structures, such as semigroups, rings, groups and Moufang loops. The study of commutativity degree of finite groups started more than half a century ago. The *commutativity degree* (or *commuting probability*) of a finite algebraic structure is defined to be the probability that two randomly chosen elements of that algebraic structure commute with each other. In fact, one measures the abelianness (or commutativeness) of a finite algebraic structure A by counting the number of pairs of elements of A that commute. Let us denote it by $Pr(A)$. Formally, we have:

$$Pr(A) = \frac{\mid \{(x,y) \in A^2 \mid xy = yx\} \mid}{\mid A \mid^2} = \frac{\sum_{x\in A} \mid C_A(x) \mid}{\mid A \mid^2},$$

where $C_A(x)$ is the centralizer of x in A. For a finite group A it has been proved that $Pr(A) = \frac{k(A)}{|A|}$ where $k(A)$ is the number of conjugacy classes of A (see [5, 7] for example).

The first surprising or fascinating known fact about $Pr(G)$, where G is a finite group, is that $Pr(G) \approx 1$ implies $Pr(G) = 1$. In other words, there is no finite group G with $\frac{5}{8} < Pr(G) < 1$. Also, it has been proved that every finite group G with $Pr(G) = \frac{5}{8}$ must be nilpotent, [5]. There are other gaps in the set:

$$\mathfrak{P}_1 = \{Pr(G) \mid G \ \ is \ \ a \ \ finite \ \ group\}.$$

For example, there is no finite group G with $\frac{7}{16} < Pr(G) < \frac{1}{2}$; however, there is a group G of the order 16 with $Pr(G) = \frac{7}{16}$ and $Pr(S_3) = \frac{1}{2}$, where S_3 is the symmetric group of degree 3. So, a natural question occurs:

"*What is the set* $\mathfrak{P}_1$ *look like?*"

This question and some others first studied in general by K.S. Joseph in 1977, [6], who proposed the following three conjectures:

(J_1) All limit points of $\mathfrak{P}_1$ are rational;

(J_2) $\mathfrak{P}_1$ is a well-ordered set by $>$;

(J_3) $\mathfrak{P}_1 \cup \{0\}$ is a closed set.

Recently, S. Eberhard in [4], has shown that conjectures J_1 and J_2 are true, and so $\mathfrak{P}_1$ is nowhere dense. In this direction, he has used the so-called *Egyptian*

Fractions and their properties. Some years ago, the speaker of this talk has shown in [1] that inspite of groups, in finite semigroups, 1 is a limit point of the set:

$$\mathfrak{P}_2 = \{Pr(S) \mid S \ \ is \ \ a \ \ finite \ \ semigroup\}.$$

He has presented an infinite class of finite non-commutative semigroups and proved that the commutativity degree of the semigroups in that class may be arbitrarily close to 1 and called this class of semigroups: *almost commutative* or *approximately abelian semigroups*. Although, D. MacHale proved in 1976, [7], that there is no finite ring R with $\frac{5}{8} < Pr(R) < 1$, however, there is a ring R of the order 8 with $Pr(R) = \frac{5}{8}$, and so the bound $\frac{5}{8}$ is the best possible. Also, the speaker has conjectured that just like groups, in finite Moufang loops, there is no finite Moufang loop M with $\frac{23}{32} < Pr(M) < 1$. Actually, for an important class of finite Moufang loops called Chain loops and its modifications, the same facts that are satisfied by groups are also true, [2]. Specially, the analogous of conjectures J_1 and J_2 are true for the class of finite Chain loops and its modifications. So, the set:

$$\mathfrak{P}_3 = \{Pr(M) \mid M \ \ is \ \ a \ \ finite \ \ Chain \ \ loop\}$$

is nowhere dense and well-ordered by $>$.

Now, since by J_2, $\mathfrak{P}_1$ is a well-ordered set, another question is:

" *What is the order type of* $\mathfrak{P}_1$*?*"

Eberhard has proved in [4] that the order type of $(\mathfrak{P}_1, >)$ is either ω^ω or ω^{ω^2} (only two posibilities). So, we deduce that the order type of $(\mathfrak{P}_3, >)$ is also either ω^ω or ω^{ω^2}. As the same way, we may ask the similar questions for finite rings and semigroups.

Also recently, the speaker has defined a new notion, the *associativity degree* (or *associating probability*) of a finite loop L, denoted by $Pas(L)$, as the probability that three (randomly chosen) elements of L associate with respect to its operation. Formally, we have:

$$Pas(L) = \frac{|\{(x, y, z) \in L^3 \mid x(yz) = (xy)z\}|}{|L^3|}.$$

Then he has tried to obtain a best upper bound for $Pas(M)$, where M is a finite non-associative *Moufang loop*. He has still shown that for the class of Chain loops, and its modifications, this best upper bound is $\frac{43}{64}$ and it is related to the commutativity degree of M, $Pr(M)$. Here is also, the conjecture is: *for any finite Moufang loop* M*,* $Pas(M) \leq \frac{43}{64}$, [3]. We know that for the commutativity degree, $Pr(L) = 1$ iff L is commutative. Here is also, $Pas(L) = 1$ iff L is associative (so is a group). Therefore, by the above facts we deduce that the set:

$$\mathfrak{P}_4 = \{Pas(M) \mid M \ \ is \ \ a \ \ finite \ \ Chain \ \ loop\}$$

is nowhere dense and well-ordered by $>$, and also, the order type of $(\mathfrak{P}_4, >)$ is either ω^ω or ω^{ω^2}. Finally, we will propose some other conjectures and questions about the Commutativity degrees in finite algebraic structures and also about associativity degrees in finite Moufang loops.

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Universal groups of intermediate growth and their invariant random subgroups

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Invariant random subgroup (abbreviated IRS) is a convenient term that stands for a probability measure on the space of subgroups of a group, invariant under the action of the group by conjugation. In the case of a countable group G the space $S(G)$ of subgroups of G is supplied with the topology induced from the Tychonoff topology on $\{0,1\}^G$ where a subgroup $H \leq G$ is identified with its characteristic function $\chi_H(g) = 1$ if $g \in H$ and 0 otherwise. The delta mass corresponding to a normal subgroup is a trivial example of an IRS, as well as the average over a finite orbit of delta masses associated with subgroups in a finite conjugacy class. Hence, we are rather interested in continuous invariant probability measures on $S(G)$. Clearly, such a measure does not necessarily exist, for example if the group only has countably many subgroups. Given a countable group G, a basic question is whether a continuous IRS exists. Ultimately one wants to describe the structure of the simplex of invariant probability measures of the topological dynamical system $(Inn(G), S(G))$ where $Inn(G)$ is the group of inner automorphisms of G acting on $S(G)$. Of particular interest are ergodic measures, i.e., the extremal points in the simplex.

A very fruitful idea in the subject belongs to Anatoly Vershik who introduced the notion of a totally non free action of a locally compact group G on a space X with invariant measure μ, i.e., an action with the property that different points $x \in X$ have different stabilizers $St_G(x)$ μ-almost surely. Then the map $St : X \to S(G)$ defined by $x \mapsto St_G(x)$ is injective μ-almost surely and the image of μ under this map is the law of an IRS on G which is continuous and ergodic whenever μ is.

Recall that, given a finitely generated group G with a system of generators S, one can consider its growth function $\gamma(n) = \gamma_{(G,S)}(n)$ which counts the number of elements of length at most n. The growth type of this function when $n \to \infty$ does not depend on the generating set S and can be polynomial, exponential or intermediate. The question of existence of groups of intermediate growth was raised by Milnor and was answered by Grigrochuk in [2]. The main construction associates with every sequence $\omega \in \Omega = \{0,1,2\}^{\mathbb{N}}$ a group G_ω generated by four involutions $a_\omega, b_\omega, c_\omega, d_\omega$ and if ω is not an eventually constant sequence, then G_ω has intermediate growth. Moreover, it was also observed in [2] that the groups G_ω fall into the class of just-infinite branch groups. A group is just infinite if it is infinite but every proper quotient is finite. A group is branch if it has a faithful level transitive action on a spherically homogeneous rooted tree with the property that rigid stabilizers of the levels of the tree are of finite index.

* Joint work with R. Grigorchuk and T. Nagnibeda

Since the groups G_ω are just-infinite, they only have countably many quotients. This raised the question of existence of groups of intermediate growth having $2^{\aleph_0}$ quotients. Our main theorem is:

Theorem. *There exists a finitely generated group of intermediate growth with* $2^{\aleph_0}$ *distinct continuous ergodic invariant random subgroups.*

The main idea is to take a suitable subset $\Lambda \subset \Omega$ of cardinality $2^{\aleph_0}$ and consider the group U_Λ (which we call the universal group associated to this family) defined as the quotient of the free group F_4 by a normal subgroup N which is the intersection of normal subgroups $N_\omega, \omega \in \Lambda$ where $G_\omega = F_4/N_\omega$. In this paper we explore this idea further by using IRS on G_ω and lift them to U_Λ deducing the main result.

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Products of Finite Groups

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Let G be a finite group and A, B proper subgroups of G. If $G = AB$, then we say that G is a factorizable group and A, B are called factors of this factorization. In this case G is also called the products of two proper subgroups A and B. The problem of which finite groups are factorizable is still an open problem.

In the book, products of groups, authored by F. De Giovanni, et al. [1], page 13, the authors raise the the question to describe all groups that have a proper factorization. Although this question is still an open problem, but by imposing some conditions on factors we are able to find factorization of a group G. This condition may be to assume that one factor is a simple group or alternating or a symmetric group, etc. In particular one can see the references [2], [3] and [4]. In this talk we survey results on factorizations of finite groups.

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Triangle-Free Commuting Conjugacy Class Graphs

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There are many ways a graph is associated with conjugacy classes of a group. In 2009, Herzog, Longobardi and Maj [1] introduced the *commuting conjugacy class graph* $\Gamma(G)$ of G associated with the non-central conjugacy classes of G. The vertices of $\Gamma(G)$ are the non-central conjugacy classes of G and two distinct vertices C and D are adjacent whenever there exist two elements $x \in C$ and $y \in D$ such that $xy = yx$. They prove, in particular that for non-abelian periodic groups G, $\Gamma(G)$ is an empty graph if and only if G is isomorphic to one of the groups S_3, D_8 or Q_8. The aim of this article is to classify all finite groups G with a triangle-free commuting conjugacy class graph. We will state the structure of all groups G with $\Gamma(G)$ is a triangle-free whenever G has odd or even order, G is non-abelian soluble or centerless non-soluble. For instance, we prove the following :

Theorem. *Let G be a finite group whose commuting conjugacy class graph $\Gamma(G)$ is a triangle-free.*

(i) *If G is a group of odd order, then $|G| = 21$ or 27 .*

(ii) *Suppose G is a group of even order which is not a 2-group. If $Z(G) \neq 1$ then G is isomorphic to D_{12} or $T_{12} = \langle a, b \big| a^4 = b^3 = 1, b^a = a^{-1}\rangle$.* **(iii)** *If G is a centerless non-soluble group, then G is isomorphic to one of the groups $PSL(2, q)$ ($q \in \{4, 7, 9\}$), $PSL(3, 4)$ or* SmallGroup(960, 11357). **(iv)** *If G is a non-abelian soluble group with $Z(G) = 1$, then G is isomorphic to one of the groups: S_3, D_{10}, A_4, S_4,* SmallGroup(72, 41), SmallGroup(192, 1023) *or* SmallGroup(192, 1025).

Note that the nth group of order m in the GAP small groups library is denote by SmallGroup(m, n).

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* Joint work with A. Mohammadian, M. Farrokhi D.G and B. Wilkens

Finite groups admitting a dihedral group of automorphisms

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Let F be a nilpotent group acted on by a group H via automorphisms and let the group G admit the semidirect product FH as a group of automorphisms so that $C_G(F) = 1$. By a well known result [1] due to Belyaev and Hartley, the solvability of G is consequence of the fixed point free action of the nilpotent group F. A lot of research, [7],[10], [11], [13], [14], [15] investigating the structure of G has been conducted in case where FH is a Frobenius group with kernel F and complement H. So the immediate question one could ask was whether the condition of being Frobenius for FH could be weakened or not. In this direction we introduced the concept of a Frobenius-like group in [8] as a generalization of Frobenius group and investigated the structure of G when the group FH is Frobenius-like [3],[4],[5],[6]. In particular, we obtained in [3] the same conclusion as in [10]; namely the nilpotent lengths of G and $C_G(H)$ are the same, when the Frobenius group FH is replaced by a Frobenius-like group under some additional assumptions. In a similar attempt in [16] Shumyatsky considered the case where FH is a dihedral group and proved the following:

Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β and let $F = \langle \alpha\beta \rangle$. (Here, $D = FH$ where $H = \langle \alpha \rangle$) Suppose that D acts on the group G by automorphisms in such a way that $C_G(F) = 1$. If $C_G(\alpha)$ and $C_G(\beta)$ are both nilpotent then G is nilpotent.

In the present paper we extend his result as follows:

Theorem. *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β and let $F = \langle \alpha\beta \rangle$. Suppose that D acts on the group G by automorphisms in such a way that $C_G(F) = 1$. Then the nilpotent length of G is equal to the maximum of the nilpotent lengths of the subgroups $C_G(\alpha)$ and $C_G(\beta)$.*

After completing the proof we became aware of the paper [2] by de Melo and seen that the above theorem follows as a corollary of the main theorem of the paper of de Melo, which is given below:

Theorem. *Let $M = FH$ be a finite group that is a product of a normal abelian subgroup F and an abelian subgroup H. Assume that all elements in $M \backslash F$ have prime order p, and F has at most one subgroup of order p. Suppose that M*

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acts on a finite group G in such a manner that $C_G(F) = 1$. Then $F_i(C_G(H)) = F_i(G) \cap C_g(H)$ for all i and $h(G) \leq h(C_G(H)) + 1$.

The proof we give relies on the investigation of D-towers in G in the sense of [17] and the following proposition which, we think, can be effectively used in similar situations.

Proposition. *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β. Suppose that D acts on a q-group Q for some prime q and let V be a kQD-module for a field k of characteristic different from q such that the group $F = \langle \alpha\beta \rangle$ acts fixed point freely on the semidirect product VQ. If $C_Q(\alpha)$ acts nontrivially on V then we have $C_V(\alpha) \neq 0$ and $Ker(C_Q(\alpha)\, on\, C_V(\alpha)) = Ker(C_Q(\alpha)\, on\, V)$.*

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Representations of Leavitt path algebras over an additive category with Krull-Schmidt

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In an additive category with Krull-Schmidt every object has a unique (up to ordering) representation as a (finite) direct sum of indecomposables. This is less restrictive than a Krull-Schmidt category where the endomorphism rings of indecomposables need to be local [K]. The category of finitely generated modules over a PID is an example. The category of unital modules of an Leavitt path algebra (over a field) is equivalent to the category of functors from a digraph (regarded as a small category where vertices are the objects and paths are the morphisms) to vector spaces satisfying an isomorphism condition by [KO, Theorem 2]. This enables us to talk about representations of an LPA (Leavitt path algebra) over an additive category without actually defining the LPA. In fact the algebra is only defined up to Morita equivalence when it exists! If Krull-Schmidt holds in the additive category then there is a classification of all representations similar to the classification of all finite dimensional representations over a field [KO, Theorem 32]. In particular there is a nonzero unital module if and only if the digraph has a maximal cycle or a maximal sink.

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On the subgroup generated by autocommutators

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It is well-known that the set of all commutators in a group is not necessarily a subgroup, see for instance the nice survey by L-C. Kappe and R.F. Morse [4]. Many authors have considered subsets of a group G related to commutators asking if they are subgroups.

Now let $(G, +)$ be an abelian group. With $g \in G$ and $\varphi \in Aut(G)$, the automorphism group of G, we define the autocommutator of g and φ as

$$[g, \varphi] = -g + g^{\alpha}.$$

We denote by

$$K^{\star}(G) = \{[g, \varphi] \mid g \in G, \varphi \in Aut(G)\}$$

the set of all autocommutators of G and, following [2], we write

$$G^{\star} = \langle K^{\star}(G) \rangle.$$

D. Garrison, L-C. Kappe and D. Yull proved in [1] that in a finite abelian group the set of autocommutators always forms a subgroup. Furthermore they found a nilpotent group of class 2 and of order 64 in which the set of all autocommutators does not form a subgroup, and they proved that it is an example of minimal order. In this talk we will discuss the relationship between $K^{\star}(G)$ and $G^{\star}$ in infinite abelian groups, as done in [3] jointly with L-C. Kappe and M. Maj.

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* Joint work with Luise-Charlotte Kappe and Mercede Maj

Recognize some structural properties of a finite group from orders of its elements

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Let G be a periodic group. The problem of obtaining information about the structure of G by looking at the orders of its elements has been considered by many authors, from many different points of view. In this talk we consider a finite group G, and we study the function on the element orders of G defined by

$$\psi(G) = \sum_{x \in G} o(x),$$

where $o(x)$ is the order of the element x.

H. Amiri, J. Amiri and M. Isaacs proved that if G has order n and C_n denotes the cyclic group of order n, then

$$\psi(G) \leq \psi(C_n),$$

and

$$\psi(G) = \psi(C_n) \quad if\ and\ only\ if \quad G \simeq C_n.$$

We discuss some results concerning the structure of the group G assuming some inequalities involving $\psi(G)$.

Some other functions on the orders of the elements of a finite group G have been recently investigated by M. Garonzi and M. Patassini

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* Joint work with Marcel Herzog and Patrizia Longobardi

A Semi-historical introduction to Leavitt path algebras

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LPAs (Leavitt Path Algebras) were defined just over a decade ago (Abrams and Aranda Pino, 2005; Ara, Moreno and Pardo, 2007) but they have roots in the works of Leavitt in the 60s focused on understanding the extent of the failure of the IBN (Invariant Basis Number) property for arbitrary rings. A ring has IBN if any two bases of a finitely generated free module have the same number of elements. Fields, division rings, commutative rings, Noetherian rings all have IBN. A classical example of a ring without IBN is the algebra of endomorphisms of a countably infinite dimensional vector space. The free module of rank 1 over this ring has bases of n elements for any positive integer n. In the early 60s Bill Leavitt asked and then answered this question: Given any $m < n$ (positive integers) is there a ring R having a free module with a basis of m elements and another basis with n elements but no bases with k elements if $k < n$ and not equal to m?

The algebras Leavitt constructed are now called the Leavitt algebras and denoted by $L(m,n)$. They are simple iff $m = 1$ and have a semi-universal property among the algebras whose free module of rank m also has a basis of n elements. That a free $L(m,n)$ module of rank m has a basis of n elements is immediate from the definition. However, to show that there are no bases of k elements ($k < n$, different from m) is essentially a question of nonstable K-theory and highly nontrivial. This was put in a broader context by George Bergman in the 70s by constructing rings R with essentially arbitrary $V(R)$ (= the monoid of isomorphism classes of finitely generated projective R-modules under direct sum).

$L(1,n)$ is also a Cohn localization (defined by P. M. Cohn in the 70s) of the noncommutative polynomial algebra in n variables. Since this algebra is the path (or quiver) algebra of the rose with n petals, with 20/20 hindsight we see a glimmer of the connection with path algebras. In fact it took three more decades and a detour through Functional Analysis for the LPAs to be defined. The C^*-algebras defined by Joachim Cuntz (1977) and later generalizations by Cuntz-Krieger, Pimsner and others led to the theory of graph C^*-algebras developed in the 90s (and still active). Now the algebra is defined by a di(rected)graph so the combinatorial properties of the graph yield corresponding algebraic properties (both for graph C^*-algebras and the LPAs). LPAs include matrix algebras, the Jacobson-Toeplitz algebra, quantum spheres and many others as well as the Leavitt algebras. Most of these algebras have IBN! The closure of an LPA with respect to an appropriate norm yields the corresponding C^*-graph algebra.

The rings $L(1,n)$ defined by Leavitt and their analytic cousins, the C^*-algebras of Cuntz are not artificial or pathological structures constructed only for the sake of providing counterexamples; for instance they implicitly come up in Signal Processing (as the algebras generated by the downsampling and upsampling operators). Moreover Leavitt's work provided important impetus for major developments in noncommutative ring theory in the 70s by Cohn, Bergman and others.

I plan to start with the basic definitions, state some fundamental results, explain the criterion for an LPA to have IBN (joint work with Muge Kanuni Er) and, if time permits, indicate the ideas involved in the recent classification of the finite dimensional representations (jointly with Ayten Koc). While LPAs are (Cohn) localizations of Path (or Quiver) Algebras whose finite dimensional representations are usually wild, the category of finite dimensional representations of LPAs turn out to be tame with a very reasonable classification of all the indecomposables and the simples. All finite dimensional quotients of LPAs are also easy to describe.

Algebraic sets with fully characteristic radicals

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Let G be a group and S be a system of group equations with coefficients in G. We denote by $\mathrm{Rad}_G(S)$ the set of all group equations which are logical consequences of S in G. In general, one can not give a deductive description of $\mathrm{Rad}_G(S)$, because it depends on the axiomatizablity of the prevariety generated by G. In this direction, any good description of the radicals is important from the algebraic geometric point of view.

In this talk, we give a necessary and sufficient condition for $\mathrm{Rad}_G(S)$ to be fully characteristic (invariant under all endomorphisms). We apply our main result to obtain connections between radicals, identities, coordinate algebras and relatively free groups. Although most of the results can be formulate in the general frame of arbitrary algebraic structures, we mainly focus on groups in what follows. As a summary, we give here some results in the case of coefficient free algebraic geometry of groups.

Let $E \subseteq G^n$ be an algebraic set (with no coefficients). Then the radical $\mathrm{Rad}(E)$ is a fully characteristic (equivalently verbal) subgroup of the free group F_n, if and only if, there exists a family $\{K_i\}$ of n-generator subgroups of G such that $E = \bigcup_i K_i^n$. As a result, we will show that if $\mathrm{Rad}_G(S)$ is a verbal subgroup of F_n, then there exists a family $\mathfrak{X}$ of n-generator subgroups of G such that $\mathrm{Rad}_G(S)$ is exactly the set of all group identities valid in $\mathfrak{X}$. We also see that under this conditions, there exists a variety $\mathbf{W}$ of groups, such that the n-generator relatively free group in $\mathbf{W}$ is the coordinate group of S. We will prove also that if G is a nilpotent group of class at most n and $E \subseteq G^n$ is an algebraic set, then $\mathrm{Rad}(E)$ is a characteristic subgroup of F_n, if and only if $E = \bigcup_i K_i^n$ for some family $\{K_i\}$ of n-generator subgroups of G.

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