Some Fascinating Features of Commutativity Degree in Finite Algebraic Structures

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In this short talk, we discuss some of fascinating and interesting features and aspects about commutativity degree in finite algebraic structures, such as semigroups, rings, groups and Moufang loops. The study of commutativity degree of finite groups started more than half a century ago. The *commutativity degree* (or *commuting probability*) of a finite algebraic structure is defined to be the probability that two randomly chosen elements of that algebraic structure commute with each other. In fact, one measures the abelianness (or commutativeness) of a finite algebraic structure A by counting the number of pairs of elements of A that commute. Let us denote it by $Pr(A)$. Formally, we have:

$$Pr(A) = \frac{\mid \{(x,y) \in A^2 \mid xy = yx\} \mid}{\mid A \mid^2} = \frac{\sum_{x\in A} \mid C_A(x) \mid}{\mid A \mid^2},$$

where $C_A(x)$ is the centralizer of x in A. For a finite group A it has been proved that $Pr(A) = \frac{k(A)}{|A|}$ where $k(A)$ is the number of conjugacy classes of A (see [5, 7] for example).

The first surprising or fascinating known fact about $Pr(G)$, where G is a finite group, is that $Pr(G) \approx 1$ implies $Pr(G) = 1$. In other words, there is no finite group G with $\frac{5}{8} < Pr(G) < 1$. Also, it has been proved that every finite group G with $Pr(G) = \frac{5}{8}$ must be nilpotent, [5]. There are other gaps in the set:

$$\mathfrak{P}_1 = \{Pr(G) \mid G \ is \ a \ finite \ group\}.$$

For example, there is no finite group G with $\frac{7}{16} < Pr(G) < \frac{1}{2}$; however, there is a group G of the order 16 with $Pr(G) = \frac{7}{16}$ and $Pr(S_3) = \frac{1}{2}$, where S_3 is the symmetric group of degree 3. So, a natural question occurs:

"*What is the set* $\mathfrak{P}_1$ *look like?*"

This question and some others first studied in general by K.S. Joseph in 1977, [6], who proposed the following three conjectures:

(J_1) All limit points of $\mathfrak{P}_1$ are rational;

(J_2) $\mathfrak{P}_1$ is a well-ordered set by $>$;

(J_3) $\mathfrak{P}_1 \cup \{0\}$ is a closed set.

Recently, S. Eberhard in [4], has shown that conjectures J_1 and J_2 are true, and so $\mathfrak{P}_1$ is nowhere dense. In this direction, he has used the so-called *Egyptian Fractions* and their properties.

Some years ago, the speaker of this talk has shown in [1] that inspite of groups, in finite semigroups, 1 is a limit point of the set:

$$\mathfrak{P}_2 = \{Pr(S) \mid S \ \ is \ \ a \ \ finite \ \ semigroup\}.$$

He has presented an infinite class of finite non-commutative semigroups and proved that the commutativity degree of the semigroups in that class may be arbitrarily close to 1 and called this class of semigroups: *almost commutative* or *approximately abelian semigroups*. Although, D. MacHale proved in 1976, [7], that there is no finite ring R with $\frac{5}{8} < Pr(R) < 1$, however, there is a ring R of the order 8 with $Pr(R) = \frac{5}{8}$, and so the bound $\frac{5}{8}$ is the best possible.

Also, the speaker has conjectured that just like groups, in finite Moufang loops, there is no finite Moufang loop M with $\frac{23}{32} < Pr(M) < 1$. Actually, for an important class of finite Moufang loops called Chain loops and its modifications, the same facts that are satisfied by groups are also true, [2]. Specially, the analogous of conjectures J_1 and J_2 are true for the class of finite Chain loops and its modifications. So, the set:

$$\mathfrak{P}_3 = \{Pr(M) \mid M \ \ is \ \ a \ \ finite \ \ Chain \ \ loop\}$$

is nowhere dense and well-ordered by $>$.

Now, since by J_2, $\mathfrak{P}_1$ is a well-ordered set, another question is:

" *What is the order type of* $\mathfrak{P}_1$*?*"

Eberhard has proved in [4] that the order type of $(\mathfrak{P}_1, >)$ is either ω^ω or ω^{ω^2} (only two posibilities). So, we deduce that the order type of $(\mathfrak{P}_3, >)$ is also either ω^ω or ω^{ω^2}. As the same way, we may ask the similar questions for finite rings and semigroups.

Also recently, the speaker has defined a new notion, the *associativity degree* (or *associating probability*) of a finite loop L, denoted by $Pas(L)$, as the probability that three (randomly chosen) elements of L associate with respect to its operation. Formally, we have:

$$Pas(L) = \frac{|\{(x, y, z) \in L^3 \mid x(yz) = (xy)z\}|}{|L^3|}.$$

Then he has tried to obtain a best upper bound for $Pas(M)$, where M is a finite non-associative *Moufang loop*. He has still shown that for the class of Chain loops, and its modifications, this best upper bound is $\frac{43}{64}$ and it is related to the commutativity degree of M, $Pr(M)$. Here is also, the conjecture is: *for any finite Moufang loop* M, $Pas(M) \leq \frac{43}{64}$, [3]. We know that for the commutativity degree, $Pr(L) = 1$ iff L is commutative. Here is also, $Pas(L) = 1$ iff L is associative (so is a group). Therefore, by the above facts we deduce that the set:

$$\mathfrak{P}_4 = \{Pas(M) \mid M \ \ is \ \ a \ \ finite \ \ Chain \ \ loop\}$$

is nowhere dense and well-ordered by $>$, and also, the order type of $(\mathfrak{P}_4, >)$ is either ω^ω or ω^{ω^2}.

Finally, we will propose some other conjectures and questions about the Commutativity degrees in finite algebraic structures and also about associativity degrees in finite Moufang loops.

References

[1] K. Ahmadidelir, C.M. Campbell, and H. Doostie, *Two classes of finite semigroups and monoids involving Lucas numbers*, Semigroup Forum, **78** (2009), 200–209.

[2] K. Ahmadidelir, *On the Commutativity Degree in Finite Moufang Loops*, to appear in *Int. J. Group Theory (IJGT)*, Articles in Press, Accepted Manuscript, Available Online from 02 March 2015.

[3] K. Ahmadidelir, *On the Associativity Degree in Finite Moufang Loops*, submitted to Int. J. Alg. & Comp., (2015).

[4] S. Eberhard, *Commuting probabilities of finite groups*, Bull. Lond. Math. Soc., **47**(5) (2015), 796–808.

[5] W.H. Gustafson, *What is the probability that two group elements commute?*, Amer. Math. Monthly, **80** (1973), 1031–1034.

[6] K.S. Joseph, *Several Conjectures on Commutativity in Aalgebraic structures*, Amer. Math. Monthly, **84** (1977), 550–551.

[7] D. MacHale, *Commutativity in finite rings*, Amer. Math. Monthly, **83** (1976), 30–32.