KAPLANSKY ZERO DIVISOR CONJECTURE ON GROUP ALGEBRAS OVER TORSION-FREE GROUPS

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Abstract. Let G be a any torsion-free group and R be an arbitrary commutative integral domain. Kaplansky's Zero Divisor conjecture states that the group ring $R[G]$ has no zero divisor, that is if $ab = 0$ for some $a, b \in R[G]$, then $a = 0$ or $b = 0$. It is known that $R[G]$ has no zero divisor with support of size at most 2. We will talk about the possible zero divisors in $R[G]$ whose supports have size 3, where R is the field $\mathbb{F}_2$ of order 2 or the ring of integers $\mathbb{Z}$. In particular we prove that if a zero divisor with support of size 3 exists in $\mathbb{Z}[G]$, then there exists a zero divisor in $\mathbb{Z}[G]$ whose support is contained in $\{-1, 1\}$.

1. Introduction and Results

A non-zero element α of a ring R is called a zero divisor if $\alpha\beta = 0$ or $\beta\alpha = 0$ for some non-zero element $\beta \in R$. A ring R is called a domain if R has no zero divisor.

Irving Kaplansky proposed the following famous question about the zero divisors of group algebras over torsion-free groups:

Question 1.1 (Problem 6 of [6])**.** Let G be an arbitrary torsion-free group and $\mathbb{F}$ be any field. Is it true that the group algebra $\mathbb{F}[G]$ a domain?

Question 1.1 is mostly known as Kaplansky Zero Divisor Conjecture. This is known to be true for any field $\mathbb{F}$ and one-sided orderable groups G [8]; for amalgamated free products G when the group ring of the subgroup over which the amalgam is formed satisfies the Ore condition [10]; supersolvable groups [5]; polycyclic-by-finite groups (see [1], [4] and [13]); elementary amenable groups [7]; one-relator groups [11]; congruence subgroups [9] and [3]; and certain hyperbolic groups [2].

Zero divisors with small support in group rings of torsion-free groups have been studied in [12]. It is fairly easy to show that $R[G]$ has no zero divisor with support of size at most 2.

Here we study zero divisors whose support are of size 3. Some of our results are the following:

Theorem 1.2. *Let G be an arbitrary torsion-free group. If a zero divisor with support of size 3 exists in $\mathbb{Z}[G]$, then there exists a zero divisor in $\mathbb{Z}[G]$ whose support is contained in $\{-1, 1\}$.*

A simple graph can be associated to a possible zero divisor with support of size 3 in the group algebra $\mathbb{F}_2[G]$ for a possible torsion-free group G. The graph is

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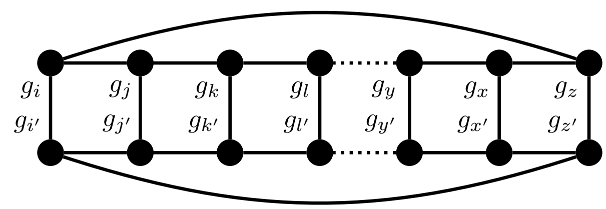
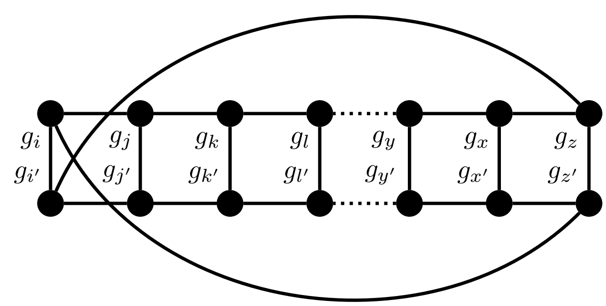
introduced in [12]. In [12] it is proved that the graph cannot have a triangle. We have proved that the graph cannot contain more than 20 other subgraphs.

Some forbidden subgraphs of the zero divisor graph



$\mathbf{L_n}$



$\mathbf{M_n}$

FIGURE 1. Two graphs which are not isomorphic to the zero divisor graph Γ

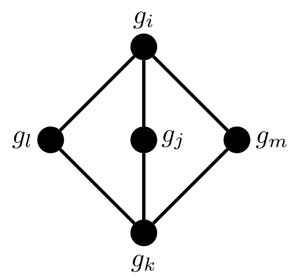


FIGURE 2. The complete bipartite graph $K_{2,3}$, a forbidden subgraph of the zero divisor graph

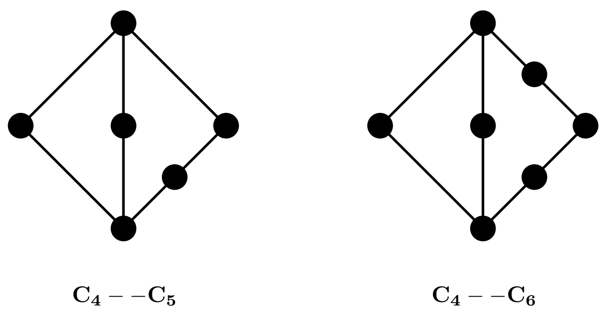


FIGURE 3. $(C_4 - -C_5)$ and $(C_4 - -C_6)$, two forbidden subgraphs of the zero divisor graph Γ

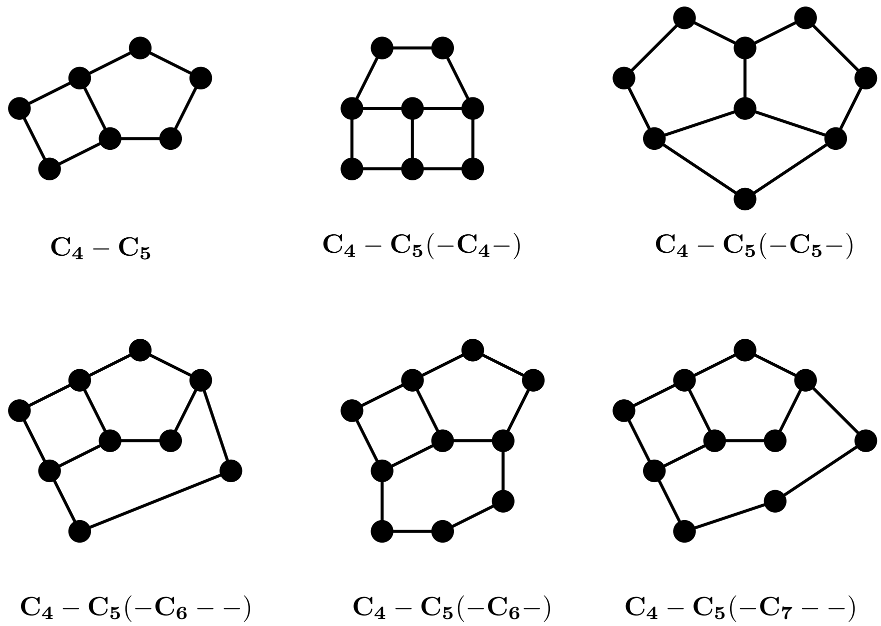


FIGURE 4. The graph $C_4 - C_5$ and some forbidden subgraphs which contains this graph

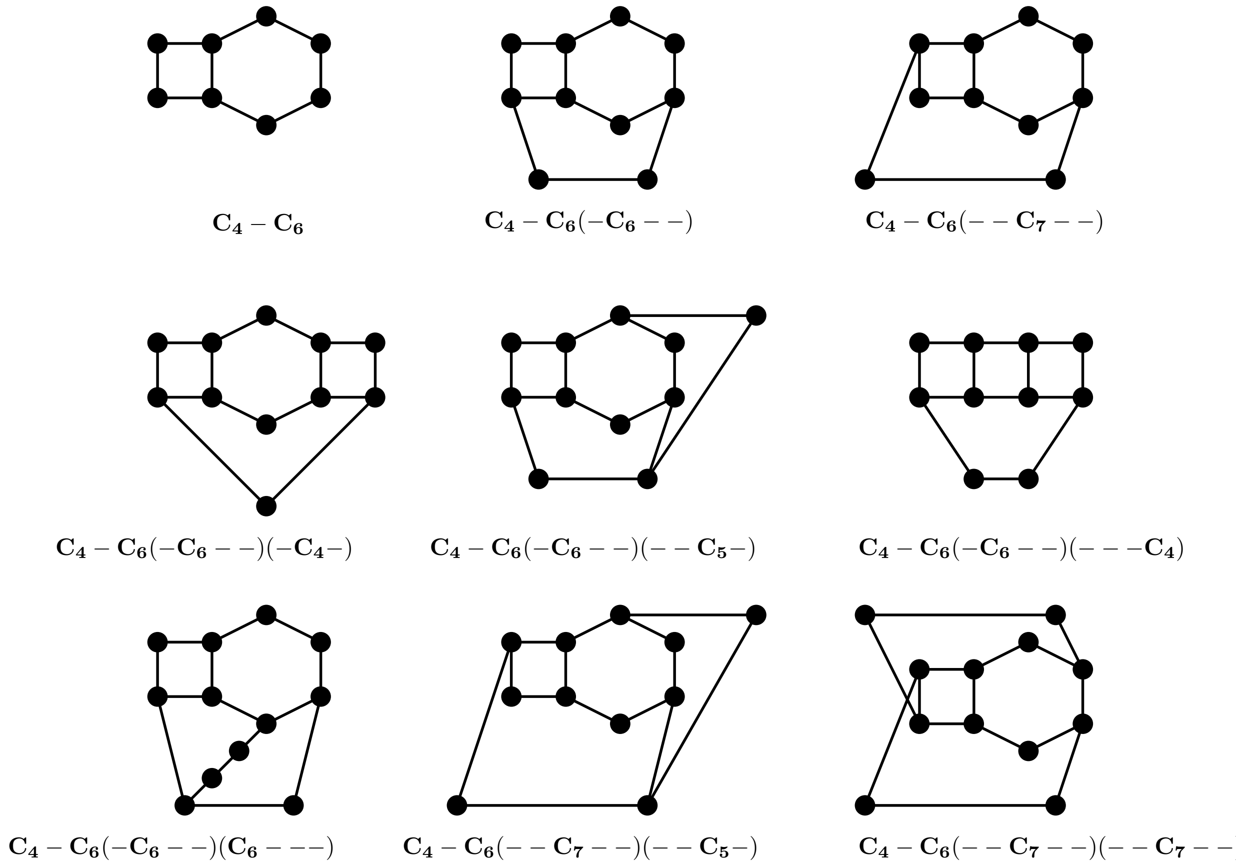
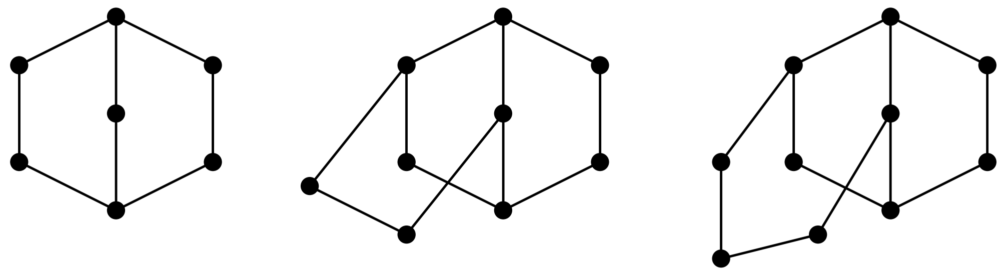
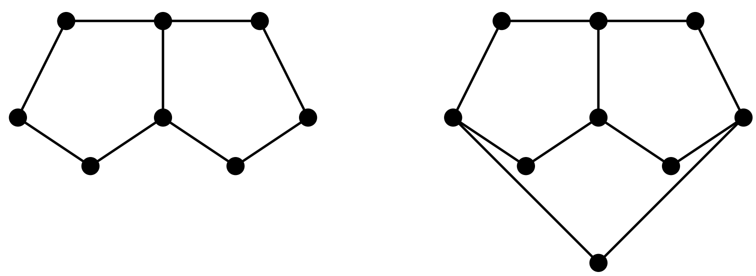


FIGURE 5. The graphs $C_4 - C_6$, $C_4 - C_6(-C_6 - -)$ and $C_4 - C_6(- - C_7 - -)$, and some forbidden subgraphs which contains one of them



$\mathbf{C_5 - - C_5}$ $\mathbf{C_5 - - C_5(- - C_5)}$ $\mathbf{C_5 - - C_5(- - C_6)}$

FIGURE 6. The graph $C_5 - -C_5$ and some forbidden subgraphs which contains it



$\mathbf{C_5 - C_5}$ $\mathbf{C_5 - C_5(- - C_6 - -)}$

FIGURE 7. The graph $C_5 - C_5$ and a forbidden subgraph which contains it

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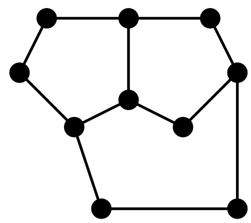
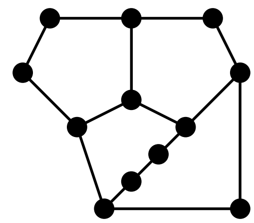
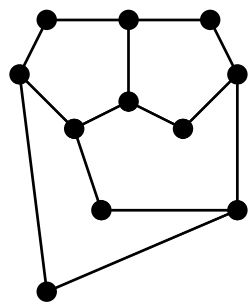


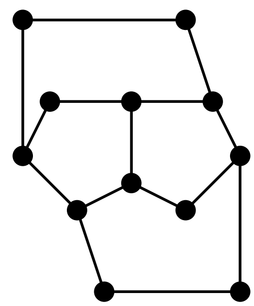
$\mathbf{C_5 - C_5(-C_6 - -)}$



$\mathbf{C_5 - C_5(-C_6 - -)(C_6 - - -)}$



$\mathbf{C_5 - C_5(-C_6 - -)(-C_5 - -)}$



$\mathbf{C_5 - C_5(-C_6 - -)(- - C_6 -)}$

FIGURE 8. The graph $C_5 - C_5(-C_6 - -)$ and some forbidden subgraphs which contains it

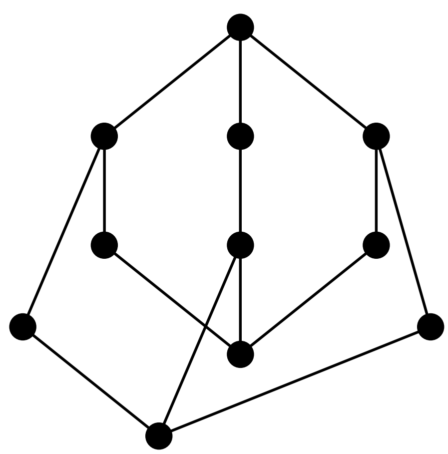


FIGURE 9. The forbidden subgraph $C6 - - - C6(C6 - - - C6)$

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